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DIFFERENTIAL ROTATION AND TURBULENT CONVECTION: A NEW REYNOLDS STRESS MODEL AND COMPARISON WITH SOLAR DATA

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ABSTRACT

In most hydrodynamic cases, the existence of a turbulent flow superimposed on a mean flow is caused by a shear instability in the latter. Boussinesq suggested the first model for the turbulent Reynolds stresses $\overline{u_i u_j}$ in the form

$$\overline{u_i u_j} = -2\nu_t S_{ij}$$

which physically implies that the mean shear S_{ij} is the cause (or source) of turbulence represented by the stress $\overline{u_i u_j}$. In the case of solar differential rotation, exactly the reverse physical process occurs: turbulence (which must pre-exist) generates a mean flow which manifests itself in the form of differential rotation. Thus, the Boussinesq model is wholly inadequate because in the solar case, cause and effect are reversed. One should envisage the sequence of cause and effect relationships as follows:

$$\text{Buoyancy} \rightarrow \text{Turbulence} \rightarrow \text{Mean Flow (Differential Rotation)}$$

where the source of turbulence has been identified with buoyancy which is present in stars for reasons unrelated to the fact that it may ultimately generate a differential rotation. An alternative way of interpreting the sequence above is by saying that small scales (buoyancy) have more energy than large scales (mean flow, differential rotation), quite contrary to most situations usually encountered in turbulence studies. Thus, the relation between buoyancy, Reynolds stresses and differential rotation must be viewed in a fundamentally different physical light from most standard hydrodynamic flows in which either the mean flow is the cause of turbulence (most laboratory and engineering cases) or both mean flow and buoyancy conspire to generate turbulence (the boundary layer of the Earth's atmosphere). Since the Boussinesq model is inadequate, one needs an alternative model for the Reynolds stresses.

We present a new dynamical model for the Reynolds stresses, convective fluxes, turbulent kinetic energy, and temperature fluctuations. The complete model requires the solution of 11 differential equations. We then introduce a set of simplifying assumptions which reduce the full dynamical model to a set of *algebraic Reynolds stress models*. We explicitly solve one of these models that entails only one differential equation. The main results are

1. Shear alone, namely the Boussinesq formula, $\overline{u_i u_j} = -2\nu_t S_{ij}$, cannot give the expected result since it describes a flow in which turbulence is generated by shear, while in the solar case shear is generated by turbulence.
2. Shear and buoyancy alone do not yield acceptable results.
3. Agreement with the data requires the nonlinear interaction between *vorticity and buoyancy*.
4. The predicted $\overline{u_\theta u_\phi}$ agrees very closely with observational data (Gilman & Howard 1984; Virtanen 1989).
5. The model predicts the magnitude and latitudinal behavior of the three components of the turbulent kinetic energy, two of which ($\overline{u_\phi^2}$ and $\overline{u_\theta^2}$) could be compared to existing data.
6. The maximum production of shear by buoyancy is predicted to occur at a latitude of $\sim 40°$.
7. The model predicts that 2.5% of the buoyant production rate is required to generate and maintain solar differential rotation.
8. The model predicts four independent anisotropic (turbulent) viscosities ν_{vv}, ν_{hh}, ν_{vh}, and ν_{hv} which depend on latitude, as well as three independent anisotropic (turbulent) conductivities, χ_{rr}, $\chi_{\phi r}$, and $\chi_{\theta r}$ which also depend on latitude (the present numerical results are restricted to radial temperature gradients).
9. The degree of anisotropy in the turbulent viscosities, measured by the parameter s, is found to depend on latitude and its value is in accordance with the empirical value of ~ 1.3.
10. The buoyancy timescale $\tau_b = [(g/H_p)(\nabla - \nabla_{ad})]^{-1/2}$ predicted by the model is in agreement with the results of stellar structure models.
11. The so-called Λ-effect is naturally (and unavoidably) predicted by the model as a result of the presence of vorticity: while shear depends only on the derivatives of Ω, vorticity also depends on Ω itself.

The overall agreement with the data is obtained with a model that is neither phenomenological nor one

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that requires a full numerical simulation, since it is algebraic in nature. The new model can play an important role in understanding the complex physics underlying the interplay between solar differential rotation and convection, as many physical processes can naturally be incorporated into the model.

Subject headings: convection — hydrodynamics — Sun: interior — Sun: rotation — turbulence

1. INTRODUCTION

The search for the physical mechanism responsible for the transport of angular momentum in a differentially rotating star has a long history that began with the quantitative analysis of Wasiutynsky (1946), Biermann (1951), Mestel (1961), Kippenhahn (1963), and others. For a detailed discussion of this subject, see Gilman (1974), Durney (1987), Rüdiger (1989), and Spruit, Nordlund, & Title (1990). Today, it seems fairly generally accepted that the angular momentum transport is contributed in large measure, if not entirely, by the Reynolds stresses $\overline{u_i u_j}$. Thus, one needs a reliable formulation of the Reynolds stresses as a function of the mean field variables such as the rotational angular velocity $\Omega(r, \theta)$, as well as of the turbulence variables, such as the turbulent kinetic energy e and its rate of dissipation, ϵ which are often combined into a turbulent viscosity $\nu_t \sim e^2/\epsilon \sim e\tau$, where τ is the characteristic time scale of turbulence. It seems fair to say that as of today no a priori derivation of the Reynolds stresses is available since all formulations are still phenomenological in nature (Kichatinov 1986, 1987, 1991; Durney 1987; for a recent review see Stix 1987, 1989). For example, in Kichatinov and Durney's models, which are the most advanced formulations presently available, the nonlinear interactions, which are at the heart of the turbulence problem, are treated with a relaxation timescale a' la mixing length theory.

Numerical simulations of the basic governing equations are emerging as an important new tool: of these, direct numerical simulations (DNS) are of limited applicability since they cannot deal with the Reynolds numbers Re and the Prandtl numbers σ typical of stars, that is Re $\sim 10^{14}$, $\sigma \sim 10^{-9}$ (Massaguer 1990). In fact, DNS calculations can only treat values like Re $\sim 10^5$ and $\sigma \sim 1$. On the other hand, large-eddy simulations (LES) can treat arbitrarily large Re, but are known to depend sensitively on the modeling of the unresolved subgrid-scales, for which there is still no completely satisfactory model. Today, most LES calculations employ a subgrid scale model first derived by Smagorinsky (1963) which was designed for shear rather than buoyancy dominated flow as is the case in stars. No subgrid model has yet been proposed which includes all the physics relevant to stellar interiors such as buoyancy, stable stratification (the overshooting region), rotation, etc. (see, however, Canuto 1993).

However, even when a physically complete subgrid-scale model will become available, LES will still remain a computationally demanding tool to be used parsimoniously, and thus, an alternative approach to the problem is of great value. Since phenomenological models, which have played an important historical role for over 30 years, seem to have exhausted their fruitfulness not ultimately because the large number of adjustable parameters severely hinders their predictability power, a more fundamental approach to the Reynolds stress is required which must be intermediate between the phenomenological models and the emerging numerical simulations.

In this paper we present a new model for $\overline{u_i u_j}$ and the flux $\overline{u_i \theta}$ that includes the three most important physical processes: *shear, vorticity, and buoyancy*. The complete model for the turbulence and mean variables contains: five differential equations for the tensor $\overline{u_i u_j}$, three differential equations for the fluxes $\overline{u_i \theta}$, one differential equation for the turbulent kinetic energy e, one differential equation for the temperature fluctuations $\overline{\theta^2}$, and one for the dissipation rate of turbulent kinetic energy ϵ. As these equations depend on the mean fields, there are, in addition three differential equations for the mean velocity U_i, one differential equation for the mean temperature T, and one differential equation for the z-component of the mean angular velocity L_z. The solution of these equations would yield all the mean and turbulent variables of interest as functions of radius r and polar angle θ.

However, even if this complete model were solved, it would yield a wealth of information which is probably in excess of what is known from observations. In addition, since the main objective of this paper is to emphasize the completeness of the physical components of the Reynolds stress model to be proposed, rather than fine-tuning the independent turbulence variables e and τ (a problem which, however, presents no a priori difficulty), we shall introduce a hierarchy of models of varying complexity. To clarify this important point, consider the functional dependence of the stresses and fluxes on the mean and turbulence variables

$$\overline{u_i u_j} = f(S_{ij}, R_{ij}, \beta_i \,|\, e, \epsilon, \overline{\theta^2}) , \tag{1}$$

$$\overline{u_i \theta} = g(S_{ij}, R_{ij}, \beta_i \,|\, e, \epsilon, \overline{\theta^2}) . \tag{2}$$

Here, the mean variables are the mean shear

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) , \tag{3a}$$

the mean vorticity

$$R_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) , \tag{3b}$$

and the mean temperature gradient

$$\beta_i = -\left[\frac{\partial T}{\partial x_i} - \left(\frac{\partial T}{\partial x_i}\right)_{ad}\right] . \tag{3c}$$

As we shall show in what follows,

$$S_{ij} = a\,\frac{\partial \Omega}{\partial r} + b\,\frac{\partial \Omega}{\partial \theta} , \tag{4}$$

$$R_{ij} = c\,\frac{\partial \Omega}{\partial r} + d\,\frac{\partial \Omega}{\partial \theta} + e\Omega , \tag{5}$$

that is, S_{ij} depends only on the derivatives of Ω while the vorticity R_{ij} depends also on Ω itself and thus is directly related to the Λ-effect (Rüdiger 1989). Also β_i represents temperature stratification: in the case of $\beta_i > 0$, stratification is unstable, the convective flux is positive and buoyancy acts like a source of turbulence; for $\beta_i < 0$, the stratification is stable, the convective flux is negative and buoyancy acts like a sink for turbulence.

As for the turbulence variables, there are four: e, the turbulent kinetic energy TKE; ϵ the rate of dissipation of TKE; $\overline{\theta^2}$, the turbulent temperature variance and ϵ_θ, its rate of dissipation. However, since the latter is usually considered to occur on a timescale $\tau_\theta = \overline{\theta^2}/\epsilon_\theta$ which is proportional to $\tau = 2e/\epsilon$, we have not written it explicitly in equations (1) and (2).

The separation in equations (1) and (2) of the dependence of $\overline{u_i u_j}$ and $\overline{u_i \theta}$ on the mean and turbulence variables, is of great conceptual importance; in fact, one of the principal results of this paper will be to show that of the three possible models

$$\overline{u_i u_j} = f(S_{ij}, 0, 0 \mid e, \epsilon, \overline{\theta^2}) , \tag{6}$$

$$\overline{u_i u_j} = f(S_{ij}, R_{ij}, 0 \mid e, \epsilon, \overline{\theta^2}) , \tag{7}$$

$$\overline{u_i u_j} = f(S_{ij}, R_{ij}, \beta_i \mid e, \epsilon, \overline{\theta^2}) , \tag{8}$$

and analogous expressions for $\overline{u_i \theta}$, only the last model yields a horizontal stress $\overline{u_\theta u_\phi}$ in accordance with observational data, as we shall show in § 8.1. In particular, we will show that without the presence of buoyancy and its interplay with R_{ij}, the functional dependence of $\overline{u_i u_j}$ on the colatitude θ is in strong disagreement with observations, irrespectively of how the turbulence variables e, ϵ, and $\overline{\theta^2}$ are treated. Based on these general results, we have considered several versions of the complete model (called M1) in order of decreasing complexity in the way the turbulence variables e, ϵ, and $\overline{\theta^2}$ are treated but with the same full dependence on the large-scale variables S_{ij}, R_{ij}, and β_j. These *algebraic Reynolds stress models* are as follows:

1. In the M2 model, $\overline{u_i u_j}$ and $\overline{u_i \theta}$ are given in algebraic form, while e, ϵ, and $\overline{\theta^2}$ are obtained from the solution of three differential equations;
2. The M3 model also has algebraic expressions for $\overline{u_i u_j}$ and $\overline{u_i \theta}$, while e and ϵ obtained from the solution of two differential equations; the temperature fluctuation $\overline{\theta^2}$ is treated algebraically;
3. The M4 model has algebraic expressions for $\overline{u_i u_j}$ and $\overline{u_i \theta}$ which are simpler than those in the M3 model, for we neglect the nonlinear contributions to the pressure strain correlation tensor and pressure-temperature correlation vector; it also has two differential equations for e and ϵ; and finally,
4. The M5, or operational model, is a simplified version of the M4 model in that the differential equation for e is reduced to an algebraic equation due to the neglect of the diffusion term. There is only one differential equation, that for ϵ.

Model M5 naturally yields a Reynolds stress $\overline{u_\theta u_\phi}$ in very good agreement with solar data (Gilman & Howard 1989; Virtanen 1989; Goode 1991); the qualitative behavior of the other components of the Reynolds stress tensor and of the convective fluxes are found to be in agreement with numerical simulation data (Gilman & Glatzmaier 1981; Glatzmaier 1984, 1985a, b, 1987; Rüdiger & Tuominen 1987, 1991; Tuominen & Rüdiger 1989; Pulkkinen et al. 1991, 1993) and phenomenological models (Durney 1991).

2. REYNOLDS STRESSES

So as not to encumber the presentation with excessive mathematical details, we relegate the derivation of the basic equations to Appendix A. Once the fluid dynamics equations are averaged over an infinite ensemble of realizations, one obtains the equations for the ensemble mean fields. If $v_i = U_i + u_i$ is the total velocity field, with U_i and u_i the mean and fluctuating fields, respectively (such that $\overline{u_i} = 0$), Reynolds averaging (denoted by an overbar) yields the following equation for U_i (we will take the flow to be incompressible, so that $\rho = 1$),

$$\frac{DU_i}{Dt} = -\left(g_i + \frac{\partial P}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial U_i}{\partial x_j} - \overline{u_i u_j}\right) + \cdots , \tag{9}$$

where g_i is the gravity vector and P is the mean pressure. This is equation (A6) where we have added the kinematic viscosity term for completeness. A similar calculation gives the equation for the mean temperature T (see eq. [A8]):

$$\frac{DT}{Dt} = \frac{\partial}{\partial x_i}\left(\chi \frac{\partial T}{\partial x_i} - \overline{u_i \theta}\right) + \cdots , \tag{10}$$

where $\chi = K/(\rho c_p)$, K being the thermal conductivity. Finally, the equation for the z-component of the mean angular momentum $L_z = U_i \eta_i$, where $\eta_i = (-y, x, 0)$, is

$$\frac{DL_z}{Dt} = -g_i \eta_i + \frac{\partial}{\partial x_j}\left[\nu \eta_i \frac{\partial U_i}{\partial x_j} - \eta_i(\overline{u_i u_j} + P\delta_{ij})\right] , \tag{11}$$

where in equations (9)–(11)

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k} . \tag{12}$$

Equations (9)–(11) for the mean fields U_i, T, and L_z are similar in form to the equations describing the laminar flow: turbulence enters through the *Reynolds stresses* and *convective fluxes*

$$\overline{u_i u_j} , \quad \overline{u_i \theta} . \tag{13}$$

In the next section, we derive the dynamic equations satisfied by the variables (13).

3. THE COMPLETE MODEL FOR THE REYNOLDS STRESSES AND CONVECTIVE FLUXES, M1

Using the Reynolds stress method to treat the fluid dynamics equations (Canuto 1992, 1993) in the presence of buoyancy and shear, one can derive the equations satisfied by the turbulent quantities. The resulting equations are given by equations (A16), (A20), (A23), and (A26), respectively, and are summarized here:

1. *Reynolds stresses* $\overline{u_i u_j}$:

$$\frac{D}{Dt}\overline{u_i u_j} + \frac{\partial}{\partial x_k}\overline{u_i u_j u_k} = -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \lambda_i \overline{u_j \theta} + \lambda_j \overline{u_i \theta} + \Pi_{ij} - \epsilon_{ij} . \tag{14}$$

2. *Turbulent kinetic energy* $e = \overline{q^2}/2$, where $q^2 = u_i u_i$:

$$\frac{De}{Dt} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{q^2 u_i} + \overline{p u_i}\right) = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} + \lambda_i \overline{u_i \theta} - \epsilon . \tag{15}$$

3. *Convective fluxes* $\overline{u_i \theta}$:

$$\frac{D}{Dt}\overline{u_i \theta} + \frac{\partial}{\partial x_j}\overline{\theta u_i u_j} = \beta_j \overline{u_i u_j} - \frac{\partial U_i}{\partial x_j}\overline{u_j \theta} + \lambda_i \overline{\theta^2} - \Pi_i^\theta + \eta_i . \tag{16}$$

4. *Temperature variance* $\overline{\theta^2}$:

$$\frac{D\overline{\theta^2}}{Dt} + \frac{\partial}{\partial x_i}\overline{u_i \theta^2} = 2\beta_i \overline{u_i \theta} + \chi \frac{\partial^2 \overline{\theta^2}}{\partial x_j^2} - 2\epsilon_\theta . \tag{17}$$

The pressure-strain correlation tensor Π_{ij} and the pressure-temperature correlation vector Π_i^θ entering equations (14) and (16) are given by equations (A27)–(A39) and equations (A41)–(A43), whereas the third-order moments appearing on the left sides of equations (14)–(17) are discussed in Appendix C.

5. *Turbulent timescales τ and τ_θ:*

$$\tau = \frac{2e}{\epsilon}, \quad \tau_\theta = \frac{\overline{\theta^2}}{\epsilon_\theta}. \tag{18}$$

6. *Dissipation rate of turbulent kinetic energy ϵ:*

$$\frac{D\epsilon}{Dt} + \frac{\partial}{\partial x_i}(\overline{\epsilon u_i}) = \frac{\epsilon}{e}(-c_1 \overline{u_i u_j} S_{ij} + c_3 \lambda_i \overline{u_i \theta}) - c_2 \frac{\epsilon^2}{e}. \tag{19}$$

The constants c will be discussed later.

7. *Dissipation rate of temperature fluctuations ϵ_θ:*

$$\epsilon_\theta = \frac{1}{2} c_\theta \overline{\theta^2} \frac{\epsilon}{e}. \tag{20}$$

Finally, equations (14)–(20) must be supplemented by equations (9)–(11) for the mean fields.

4. THE PHYSICS OF THE MODEL

Before discussing the solution of equations (14)–(20), it is important to discuss their physical content. Consider, for example, the equation for the turbulent kinetic energy e, equation (15): the first term on the left corresponds to the time dependence $\partial e/\partial t$ plus the advection term $U_j \partial e/\partial x_j$; the next term corresponds to the "diffusion" of e; in fact, it is the divergence of the flux of kinetic energy (a third-order moment):

$$F_i^{\rm ke} = \tfrac{1}{2}\overline{q^2 u_i}. \tag{21a}$$

This term is often referred to as a diffusion term because a widely used approximation, based on the use of an analogy with kinetic theory, suggests that

$$\overline{q^2 u_i} = -D_t \frac{\partial \overline{q^2}}{\partial x_i}, \tag{21b}$$

which assumes that the kinetic energy flux occurs predominantly along the gradient of the kinetic energy q^2; D_t is a "turbulent diffusion" coefficient, usually taken as the product of a mixing-length times an rms velocity. Introducing equation (21b) into equation (15), the second term assumes the form of a diffusion term. In this paper we shall not make use of equation (21b) because in turbulent convective regimes it has proven to be incorrect (Canuto 1992). Rather, in Appendix C we derive the dynamic equations for the third-order moments and propose a procedure to solve them. This will also help to highlight the limitations of the down-gradient approximation (eq. [21b]).

In most flows, the first term on the right side of equation (15) is a source of kinetic energy stemming from the interaction of the Reynolds stresses $\overline{u_i u_j}$ with the shear S_{ij}: energy is extracted from the mean flow and fed into turbulence. In the approximation first suggested by Boussinesq (1877), one writes

$$\overline{u_i u_j} = -2\nu_t S_{ij}, \tag{22a}$$

so that, with $S^2 \equiv 2S_{ij}S_{ij}$,

$$-\overline{u_i u_j} S_{ij} = \nu_t S^2. \tag{22b}$$

Since $\nu_t \sim e\tau$, the right-hand side of equation (22b) can also be written as

$$e\tau S^2 \sim \frac{e}{T_s}\frac{\tau}{T_s}, \tag{22c}$$

i.e., the rate of generation of kinetic energy $\partial e/\partial t$ is proportional to e/T_s, where T_s is a characteristic timescale of the shear, multiplied by the efficiency factor τ/T_s representing the degree of synchronization between the internal frequency of turbulence $1/\tau$ and the external "beating frequency" $1/T_s$.

In the case of unstable stratification, the next term in equation (15), $\lambda_i \overline{u_i \theta}$, is positive, and is thus also a source. Here too, if we assume the simple approximation (in analogy with eq. [22a])

$$\overline{u_i \theta} = \chi_t \beta_i, \tag{22d}$$

where β_i is defined in equation (3c), we obtain

$$\lambda_i \overline{u_i \theta} = \chi_t \tau_b^{-2}, \tag{22e}$$

where

$$\tau_b^{-1} = (\alpha g_i \beta_i)^{1/2} \tag{22f}$$

is the buoyancy frequency, and α is the volume expansion coefficient. Since $\chi_t \propto e\tau$, we obtain in analogy to equation (22c)

$$e\tau |N^2| \propto \frac{e}{T_b}\frac{\tau}{T_b}, \tag{22g}$$

where τ/T_b is the efficiency factor due to buoyancy.

Finally, the last term in equation (15), ϵ, represents the rate of dissipation of kinetic energy. It must be stressed that, although energy is dissipated at the smallest scales where viscosity is most effective, ϵ is not determined by the kinematic viscosity; the latter only determines the length scale at which dissipation occurs, while the amount of energy to be dissipated is determined by the large scales, i.e., ultimately by the source itself. Thus, the value of ϵ is not governed by the small scales or by the magnitude of the viscosity. This is a direct consequence of the fact that the nonlinear interactions conserve energy and thus, the energy (or power) fed to the system at the largest scales cascades unchanged (in magnitude) to the smallest scales where finally it is dissipated into heat.

To further clarify this point, integrate equation (15) over all space to obtain (the volume average is represented by angle brackets)

$$\langle -\overline{u_i u_j} S_{ij}\rangle + \lambda_i \langle \overline{u_i \theta}\rangle = \langle \epsilon \rangle \tag{23a}$$

or

$$P = \langle \epsilon \rangle, \tag{23b}$$

which expresses the fact that production equals dissipation globally. With this interpretation, equation (15) can be written schematically as

$$\frac{\partial e}{\partial t} + \text{advection} = \text{diffusion} + \text{production} - \text{dissipation}. \tag{23c}$$

A similar interpretation holds for equations (14), (16), and (17).

As one can notice, the equations for $\overline{u_i u_j}$ and $\overline{u_i \theta}$ contain two additional terms, namely the pressure-strain and pressure-temperature correlations Π_{ij} and Π_i^θ. These terms, the construction of which is one of the most difficult parts of the entire problem, must first be understood in their physical content. Pressure forces do not exchange energy among eddies in spite of representing nonlinear interactions. Rather, given an eddy having a specific size and energy, pressure forces tend to equalize the energy content of each of its components, and thus maintain isotropy. Thus, one would expect that in the case

$i = j = 3$, the first contribution of Π_{33} in equation (14) would be of the form

$$\frac{Dw^2}{Dt} + \cdots = -\frac{1}{\tau}\left(w^2 - \frac{1}{3}q^2\right) + \cdots , \quad (24a)$$

that is, if one begins with a value of w^2 larger than the equipartition value $q^2/3$, the Π_{33} terms act like a sink and decreases the value of w^2; on the other hand, if w^2 is too small with respect to $q^2/3$, Π_{33} acts like a source and increases w^2. Based on these physical considerations, Rotta (1951) first proposed that Π_{ij} should contain a term of the form

$$\Pi_{ij} \propto \tau^{-1} b_{ij} , \quad (24b)$$

called the "return-to-isotropy," where the anisotropy tensor is

$$b_{ij} = \overline{u_i u_j} - \tfrac{1}{3}q^2\delta_{ij} . \quad (24c)$$

That is, the restoring force provided by Π_{ij} is assumed to be proportional to the degree of anisotropy itself, and thus the linear dependence of Π_{ij} on b_{ij}. Since the pioneering work of Rotta, the pressure-strain and pressure-temperature correlations have been the subject of several studies and the topic is still being studied today (Speziale, Gatski, & Sarkar 1992), although there seems to be a fair consensus that the most relevant features have been captured by the latest formulations, which are given in equations (A27) and (A41):

$$\Pi_{ij} = 2c_4\tau^{-1}b_{ij} - \frac{4}{5}eS_{ij} - \alpha_1\Sigma_{ij} - \alpha_2 Z_{ij} + (1-\beta_5)B_{ij} + \frac{\partial}{\partial x_j}\overline{pu_i} + \frac{\partial}{\partial x_i}\overline{pu_j} + \Pi_{ij}(\mathrm{NL}) , \quad (25a)$$

$$\Pi_i^\theta = f_1\tau^{-1}\overline{u_i\theta} + \gamma_1\lambda_i\overline{\theta^2} - \frac{3}{4}\alpha_3\left(S_{ij} + \frac{5}{3}R_{ij}\right)\overline{u_j\theta} + \frac{\partial}{\partial x_i}\overline{p\theta} + \Pi_i^\theta(\mathrm{NL}) , \quad (25b)$$

where the detailed expressions for the nonlinear contributions $\Pi_{ij}(\mathrm{NL})$ and $\Pi_i^\theta(\mathrm{NL})$ are given in equations (A27b)–(A29) and (A41a)–(A43). To gain physical insight into equation (25a), consider the trace of Π_{ij}, i.e.,

$$\Pi_{ii} = 2\frac{\partial}{\partial x_i}\overline{pu_i} \sim \frac{\partial}{\partial x_i}F_i^{\mathrm{ke}} , \quad (25c)$$

where the last expression follows from the fact that the pressure is proportional to the kinetic energy. Thus, the diagonal part of Π_{ij} has the same physical interpretation as the diffusion term, which explains its presence in equation (15).

Next, consider the terms Σ_{ij} and Z_{ij}. Using equations (14) and (15) to construct the differential equation for the anisotropy tensor b_{ij} defined in equation (24c), and keeping only the terms which are relevant to the present discussion, we derive after some simple manipulations

$$\frac{Db_{ij}}{Dt} + \cdots = -(\Sigma_{ij} + Z_{ij}) - \left(\Pi_{ij} - \frac{1}{3}\Pi_{kk}\delta_{ij}\right) + \cdots , \quad (26a)$$

where the first two production terms in equation (14) are now represented by the traceless tensors

$$\Sigma_{ij} = S_{ik}b_{kj} + S_{jk}b_{ik} - \tfrac{2}{3}\delta_{ij}S_{km}b_{km} , \quad (26b)$$

$$Z_{ij} = R_{ik}b_{kj} + R_{jk}b_{ik} - \tfrac{2}{3}\delta_{ij}R_{km}b_{km} . \quad (26c)$$

In equation (26a), the combined action of Σ_{ij} and Z_{ij} represents production of b_{ij} which is anisotropic since, for example, the production of b_{11} is contributed not only by diagonal interaction terms such as $b_{11}S_{11}$, but also by off-diagonal interaction terms such as

$$b_{12}S_{12} + b_{13}S_{13} \quad \text{and} \quad b_{12}R_{12} + b_{13}R_{13} , \quad (26d)$$

which is particularly important in the case of differential rotation, since as we have already stated, the tensor S_{ij} contains only derivatives of Ω, while R_{ij} entails Ω itself, thus contributing to the well-known Λ-effect (§ 9). Since, as we have discussed, pressure forces tend to modulate anisotropies in general, it is clear that the presence of Π_{ij} in equation (26a) will tend to modulate the production term, the net result being that using equation (25a), equation (26a) becomes

$$\frac{Db_{ij}}{Dt} + \cdots = -(1-\alpha_1)\Sigma_{ij} - (1-\alpha_2)Z_{ij} + \cdots , \quad (26e)$$

so that the final result is that the anisotropic production Σ_{ij} and Z_{ij} are reduced by the fractions $(1-\alpha_1)$ and $(1-\alpha_2)$, respectively. A similar interpretation holds for the pressure-temperature correlation.

5. ALGEBRAIC REYNOLDS STRESS MODELS (A-RSM)

The solution of the full model, in conjunction with a stellar structure code, would yield all variables that are important for the study of the interplay between convection and rotation and which Schatzman (1991) has repeatedly called attention to. However, before one tackles the whole problem, one must show that the model passes a direct test: the correct prediction of the Reynolds stresses at the surface of the Sun for which we have measurements. The model we use has no adjustable parameters and therefore we have no way of forcing the outcome, a property that we consider to be a significant strength. We will assume that the mean velocity and temperature fields are given, and we will consider the solution of the following *algebraic Reynolds stress models*, the derivation of which from the complete Reynolds stress model is given in Appendix B:

5.1. *Model 2*

Reynolds stresses:

$$Ab_{ij} = -\tfrac{8}{15}\tau e S_{ij} + \beta_5\tau B_{ij} - (1-\alpha_1)\tau\Sigma_{ij} - (1-\alpha_2)\tau Z_{ij} - \tau\Pi_{ij}(\mathrm{NL}) , \quad (27a)$$

where

$$A \equiv A_0 + A_1\frac{P}{\epsilon} , \quad A_0 \equiv 2c_4^* + 2c_2 - 4 + \left(\frac{\tau}{S_*}\right)\frac{DS_*}{Dt} , \quad A_1 \equiv 4 - 2c_1^* , \quad (27b)$$

$$c_1^* \equiv \frac{1}{P}(c_1P_s + c_3P_b) , \quad P = P_s + P_b , \quad (27c)$$

and where P_s and P_b are the production rates of shear and buoyancy defined as

$$P_s = -b_{ij}S_{ij} , \quad P_b = \lambda_i\overline{u_i\theta} . \quad (27d)$$

The constants are given in Appendix D.

Convective fluxes $\overline{u_i\theta}$:

$$A_{ik}\overline{u_k\theta} = \left(\frac{2}{3}e\tau\delta_{ij} + \tau b_{ij}\right)\beta_j + (1-\gamma_1)\tau\lambda_i\overline{\theta^2} - \tau\Pi_i^\theta(\mathrm{NL}) + \frac{1}{2}\chi\tau\frac{\partial^2}{\partial x_j^2}\overline{u_i\theta} , \quad (28a)$$

where the matrix A_{ik} is given by

$$A_{ik} \equiv (f_1 + B)\delta_{ik} + (1 - \tfrac{3}{4}\alpha_3)\tau S_{ik} + (1 - \tfrac{5}{4}\alpha_3)\tau R_{ik} . \quad (28b)$$

Thus, the complete Model 2 is given by equations (27)–(28) together with the three differential equations (15), (17) and (19) for e, $\overline{\theta^2}$, and ϵ.

5.2. *Model 3*

Model 2 can be further simplified by changing equation (17) from a prognostic equation to a diagnostic equation by neglecting the time derivative and the third-order moments: using equation (20), we obtain

$$\overline{\theta^2} = C_*^{-1}\tau\beta_i \overline{u_i\theta} , \quad (29a)$$

where

$$C_* \equiv c_\theta(1 + Pe^{-1}c_\epsilon/c_\theta) , \quad (29b)$$

and Pe is the Peclet number defined in equation (B14). Thus, equation (28a) becomes

$$A_{ik}\overline{u_k\theta} = \left(\frac{2}{3} e\tau\delta_{ij} + \tau b_{ij}\right)\beta_j - \tau\Pi_i^\theta(\text{NL}) + \frac{1}{2}\chi\tau \frac{\partial^2}{\partial x_j^2}\overline{u_i\theta} , \quad (30a)$$

where the matrix A_{ik} is now given by

$$A_{ik} \equiv (f_1 + B)\delta_{ik} - C_*^{-1}(1 - \gamma_1)\tau^2\lambda_i\beta_k + (1 - \tfrac{3}{4}\alpha_3)\tau S_{ik} + (1 - \tfrac{5}{4}\alpha_3)\tau R_{ik} . \quad (30b)$$

Thus, the complete Model 3 is given by equations (27) and (30) together with the two differential equations (15) and (19) for e and ϵ.

5.3. *Model 4*

In this model we shall neglect the nonlinear contributions $\Pi_{ij}(\text{NL})$ and $\Pi_i^\theta(\text{NL})$ to equations (27a) and (30a). Furthermore, we assume that the molecular dissipation terms can be neglected and that the Peclet number is very large so that $C_* = c_\theta$. With this set of approximations, equations (27a) and (30a) simplify to:

Reynolds stresses b_{ij}:

$$Ab_{ij} = -\tfrac{8}{15}\tau e S_{ij} + \beta_5\tau B_{ij} - (1 - \alpha_1)\tau\Sigma_{ij} - (1 - \alpha_2)\tau Z_{ij} \quad (31a)$$

with A given by equation (27b), and

Convective fluxes $\overline{u_i\theta}$:

$$A_{ik}\overline{u_k\theta} = (\tfrac{2}{3}e\tau\delta_{ij} + \tau b_{ij})\beta_j , \quad (31b)$$

with A_{ik} given by equation (30b). Thus, Model 4 is given by equation (31) plus two differential equations (15) and (19) for e and ϵ.

6. THE INCORRECTNESS OF THE BOUSSINESQ MODEL

If it is assumed that the buoyancy term B_{ij} and the anisotropic production terms Σ_{ij} and Z_{ij} do not contribute to b_{ij}, equation (31a) reduces to

$$b_{ij} = -2\nu_t S_{ij} , \quad (32a)$$

where the eddy viscosity ν_t is given by

$$\nu_t = \frac{4}{15A} e\tau . \quad (32b)$$

Similarly, if we assume that shear, temperature fluctuations, and anisotropic buoyancy production $b_{ij}\beta_j$ do not contribute to the flux $\overline{u_i\theta}$, equation (31b) reduces to

$$\overline{u_i\theta} = \beta_i\chi_t , \quad (33a)$$

where the eddy conductivity χ_t is given by

$$\chi_t = \sigma_t^{-1}\nu_t , \quad \sigma_t = \frac{2}{15A}(f_1 + B) . \quad (33b)$$

Here, the turbulent Prandtl number σ_t is a constant. Even with these drastic approximations, which have little physical justification, the full model M1 has only been formally simplified since the two turbulence variables

$$e, \tau \quad \text{or} \quad e, \epsilon \quad (34a)$$

remain to be calculated from the differential equations (15) and (19). It is also of little use to combine e and ϵ into the eddy viscosity

$$\nu_t \sim e\tau \sim \frac{e^2}{\epsilon} \sim e^{1/2}l , \quad (34b)$$

since ν_t is still the product of two independent variables, the kinetic energy e and the rate of kinetic energy dissipation ϵ. As a matter of fact, the approximations made to arrive at equations (32a) and (33a) have been implemented on the "wrong" variables, since the net result is actually equation (6) which does not contain the physically correct dependence on the "large-scale variables" S_{ij}, R_{ij}, and β_j, the proper inclusion of which is essential for the correct behavior of the Reynolds stresses (§ 8.1). This can be seen as follows. In spherical coordinates, equation (32a) becomes

$$\overline{u_r u_\phi} = -\nu_t r \sin\theta \frac{\partial\Omega}{\partial r} , \quad \overline{u_\theta u_\phi} = -\nu_t \sin\theta \frac{\partial\Omega}{\partial\theta} . \quad (35a)$$

In the case of the Sun, Ω increases toward the equator, and thus $\sin\theta\, \partial\Omega/\partial\theta > 0$ in the northern hemisphere (N) and $\sin\theta\, \partial\Omega/\partial\theta < 0$ in the southern hemisphere (S), so that the second of equations (35a) implies that

$$\overline{u_\theta u_\phi} < 0 \text{ (N)} , \quad \overline{u_\theta u_\phi} > 0 \text{ (S)} , \quad (35b)$$

while observational data, Figures 1 and 15, indicate the opposite (Ward 1965; Gilman & Howard 1984; Rüdiger 1989; Pulkkinen et al. 1993). As one can see, the improper behavior of equation (35b) can hardly be expected to be remedied by the θ-dependence of ν_t.

Furthermore, equation (32a) requires that the principal axes of the two tensors b_{ij} (representing turbulence) and S_{ij} (representing the mean flow) be aligned: this is true only for the case of pure strain but not for flows with mean vorticity. For three-dimensional flows in general, the measured flow distribution can be predicted only by choosing different viscosities for each stress component. Indeed, the most complete derivations of equation (32a) indicate the presence of nonisotropic terms which break the "alignment assumption" and which are responsible for the appearance of the terms Σ_{ij} and Z_{ij}.

Finally, equation (33a) contradicts a well-known phenomenon observed in laboratory, atmospheric, and numerically simulated turbulence, namely the fact that stably stratified flows exhibit both a positive temperature gradient and a positive flux

$$\partial T/\partial z > 0 , \quad \overline{w\theta} > 0 , \quad (36)$$

which is known as the counter-gradient phenomenon (Priestley & Swinbank 1947; Deardorff 1966; Schumann 1987). It can be shown that one of the terms that contribute to the counter-gradient phenomenon and which is missing in equation (33a) is the temperature variance $\overline{\theta^2}$ (Canuto 1992, eq. [80]).

7. MODEL M5 (OPERATIONAL MODEL): APPLICATION TO THE SUN

The algebraic Reynolds stress model we shall solve in this paper is summarized as follows:

Reynolds stresses:

$$Ab_{ij} = -\tfrac{8}{15}e\tau S_{ij} + \beta_5 \tau B_{ij} - (1-\alpha_1)\tau\Sigma_{ij} - (1-\alpha_2)\tau Z_{ij}\,, \tag{37}$$

with A given by equation (27b).

Convective fluxes:

$$A_{ik}\overline{u_k\theta} = (\tfrac{2}{3}e\tau\delta_{ij} + \tau b_{ij})\beta_j\,, \tag{38a}$$

with

$$A_{ik} \equiv (f_1 + B)\delta_{ik} - C_*^{-1}(1-\gamma_1)\tau^2\lambda_i\beta_k + (1-\tfrac{3}{4}\alpha_3)\tau S_{ik} + (1-\tfrac{5}{4}\alpha_3)\tau R_{ik}\,. \tag{38b}$$

Turbulent kinetic energy, e:

$$\frac{\partial e}{\partial t} + D_f(e) = -2\overline{u_r u_\phi}S_{r\phi} - 2\overline{u_\theta u_\phi}S_{\theta\phi} + \lambda_r\overline{u_r\theta} - \epsilon\,, \tag{39a}$$

with

$$D_f(e) \equiv -\frac{4}{r^2}\frac{\partial}{\partial r}[r^2(\overline{eu_r} + \overline{pu_r})] - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta(\overline{eu_\theta} + \overline{pu_\theta})]\,. \tag{39b}$$

In this model we shall neglect the left-hand side of equation (39a) and thus equation (39a) can be rewritten as production equals dissipation

$$P_s + P_b = \epsilon\,, \tag{40a}$$

where the production P_s (by shear) and P_b (by buoyancy) are defined as

$$P_s \equiv -2(\overline{u_r u_\phi}S_{r\phi} + \overline{u_\theta u_\phi}S_{\theta\phi})\,, \tag{40b}$$

$$P_b \equiv \lambda_r\overline{u_r\theta}\,. \tag{40c}$$

Since we assume that the only nonvanishing component of U_i is

$$U_\phi = \Omega(r,\theta)r\sin\theta\,, \tag{41a}$$

the only nonvanishing components of the mean strain S_{ij} and vorticity tensors R_{ij} are given by (the problem is axisymmetric and so there is no dependence on the azimuthal angle ϕ)

$$S_{r\phi} = r\sin\theta\frac{\partial\Omega}{\partial r}\,,\quad S_{\theta\phi} = \sin\theta\frac{\partial\Omega}{\partial\theta} \tag{41b}$$

$$R_{r\phi} = -2\Omega\sin\theta - S_{r\phi}\,,\quad R_{\theta\phi} = -2\Omega\cos\theta - S_{\theta\phi}\,. \tag{41c}$$

Furthermore, we assume $\partial\Omega/\partial r = 0$, which is in reasonable accord with helioseismological studies indicating that the variation of the angular velocity in the radial direction is approximately 10% at the equator and less elsewhere (Dziembowski, Goode, & Libbrecht 1989); the same assumption was also made in the simulation work of Pulkkinen et al. (1993). Thus, $S_{r\phi} = 0$ and the production by shear P_s becomes

$$P_s = -2\overline{u_\theta u_\phi}S_{\theta\phi}\,. \tag{42a}$$

Since $\overline{u_\theta u_\phi}$ and $S_{\theta\phi}$ have observationally the same sign in both hemispheres, P_s is always negative: this means that P_s is a negative production of kinetic energy or, more exactly, a sink, which implies that turbulent kinetic energy is feeding the mean flow. The chain of physical processes is therefore

$$\text{buoyancy} \rightarrow \text{turbulence} \rightarrow \text{Reynolds stresses}\,. \tag{42b}$$

Buoyancy, generated from an independent source, creates turbulent kinetic energy (TKE) which in turn creates the Reynolds stresses responsible for setting up and maintaining differential rotation. Clearly, one expects that only a fraction of the TKE generated by buoyancy goes into creating differential rotation and thus

$$P_b \gg |P_s|\,, \tag{42c}$$

a relation that we shall discuss quantitatively in § 8.7. Physically, equation (40a) can be rewritten as

$$P_b = |P_s| + \epsilon\,, \tag{42d}$$

which means that the TKE produced by buoyancy goes partly to feed differential rotation and partly is dissipated into heat via ϵ.

As we have assumed that there is no meridional circulation, we will also assume that the only nonvanishing component of β_i is the radial component β_r. There is in principle no conceptual difficulty to include β_θ and β_ϕ. This, however, would significantly complicate the solution of the model since it would require the solution of equation (10) for the mean temperature.

In principle, the angular velocity $\Omega(r,\theta)$ is obtained by solving the angular momentum conservation equation (11), which in the stationary case becomes

$$\frac{1}{r^2}\sin\theta\frac{\partial}{\partial r}[r^3(\overline{u_r u_\phi} + U_r U_\phi)] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[\sin^2\theta(\overline{u_\theta u_\phi} + U_\theta U_\phi)] = 0\,, \tag{43a}$$

together with equation (9) for the mean velocity. However, in order to concentrate our efforts on comparing the resulting Reynolds stresses and convective fluxes with both observational and numerical simulation data, in this first application of our method we shall assume that $\Omega(r,\theta)$ is given by observational data. Specifically, we take (Howard et al. 1983)

$$\Omega(\theta) = \Omega_0(1 + a\cos^2\theta + b\cos^4\theta)\,, \tag{43b}$$

$$\Omega_0 = 2.87\ \mu\text{rad}\,,\quad a = -0.12\,,\quad b = -0.17\,. \tag{43c}$$

Next, consider the equation for ϵ, the dissipation rate of TKE. In spherical coordinates and for the stationary case, equation (19) will be taken as (R is the Sun's radius)

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\overline{\epsilon u_\theta}) = \epsilon e^{-1}(c_1P_s + c_3P_b) - c_2\epsilon^2 e^{-1}\,. \tag{44a}$$

If the diffusion term is treated with a down-gradient approximation

$$\overline{\epsilon u_i} \equiv -\nu_t\frac{\partial\epsilon}{\partial x_i}\,,\quad \nu_t = C_\mu\frac{e^2}{\epsilon} = \frac{1}{4}C_\mu\epsilon\tau^2\,, \tag{44b}$$

where a typical value is $C_\mu = 0.09$ (Rodi 1984) (for the value of C_μ used in this paper, see § 8.8), we obtain, using equation (40a),

$$\frac{\partial}{\partial \theta}\left(\sin \theta \epsilon \tau^2 \frac{\partial \epsilon}{\partial \theta}\right) = \Gamma \sin \theta \frac{\epsilon}{\tau}\left(1 + \frac{cP_s}{\epsilon}\right), \qquad (44c)$$

with

$$\Gamma \equiv \frac{8}{C_\mu}(c_2 - c_3)R^2 , \quad c \equiv (c_1 - c_3)(c_3 - c_2)^{-1} . \qquad (44d)$$

To solve equation (44c), we need two boundary conditions. The first is given by assuming that ϵ is symmetric about the equator, which implies $\partial\epsilon/\partial\theta = 0$ at the equator. The second is given by noting that ϵ is finite at the poles, which requires that $\partial\epsilon/\partial\theta = 0$ at the poles. We found that in order to satisfy the above conditions, and given the standard values $c_2 = 1.83$ and $c_1 = 1.44$, the value of c_3 was constrained to a very narrow margin around $c_3 = 1.825$. The solution of equation (44c) is thus written as

$$\epsilon(\theta) = \epsilon_* F(\theta) , \qquad (45a)$$

$$\epsilon_* \equiv \frac{8}{C_\mu}(c_2 - c_3)R^2\Omega_0^3 . \qquad (45b)$$

The TKE, solution of equation (40a), will be written as

$$e(\theta) = e_* E(\theta) , \qquad (45c)$$

where

$$e_* = \tfrac{1}{2}\epsilon_* \Omega_0^{-1} . \qquad (45d)$$

Analogously, the turbulent viscosity ν_t will be computed from equation (44b) in units of

$$\nu_t^* \equiv \tfrac{1}{4}C_\mu \epsilon_* \Omega_0^{-2} . \qquad (46a)$$

The mixing-length l will be computed from

$$l = \epsilon^{-1}e^{3/2} , \qquad (46b)$$

in units of

$$l_* = \epsilon_*^{-1}e_*^{3/2} . \qquad (46c)$$

8. SOLUTION OF MODEL 5 AND COMPARISON WITH THE DATA

Consider equation (37), which we shall write in matrix form as

$$\boldsymbol{b} = \gamma_1 e\tau \boldsymbol{S} + \gamma_2 \tau \boldsymbol{\Sigma}(\boldsymbol{b}) + \gamma_3 \tau \boldsymbol{Z}(\boldsymbol{b}) + \gamma_4 \tau \boldsymbol{B}(\boldsymbol{b}) . \qquad (47)$$

Since $\boldsymbol{\Sigma}$ and $\boldsymbol{Z}$ depend linearly on $\boldsymbol{b}$, the equation for the Reynolds stress is also linear in $\boldsymbol{b}$. This equation can be inverted algebraically, and the solution expressed in the analytical form

$$\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{\beta} \,|\, e, \tau) . \qquad (48)$$

The explicit solution is a very complicated function of θ and of little practical value for theoretical considerations, so we have not written it explicitly. It may be of interest to note that in the special case $\gamma_2 = \gamma_4 = 0$, the solution of equation (47) has been constructed using Pope's method, with a result for b_{ij} of manageable complexity (Taulbee 1992). The lack of buoyancy, however, makes this solution of interest to engineering, rather than astrophysical problems.

8.1. *Study of the Complete Model and Comparison to Solar Data*

To show that the operational model M5 contains the necessary physics to reproduce observational data both qualitatively and quantitatively, we have analyzed the contribution of each of the terms in equation (47), by considering the following five models of increasing completeness:

1. $\boldsymbol{b} = \gamma_1 e\tau \boldsymbol{S}$;
2. $\boldsymbol{b} = \gamma_1 e\tau \boldsymbol{S} + \gamma_2 \tau \boldsymbol{\Sigma}$;
3. $\boldsymbol{b} = \gamma_1 e\tau \boldsymbol{S} + \gamma_2 \tau \boldsymbol{\Sigma} + \gamma_3 \tau \boldsymbol{Z}$;
4. $\boldsymbol{b} = \gamma_1 e\tau \boldsymbol{S} + \gamma_2 \tau \boldsymbol{\Sigma} + \gamma_4 \tau \boldsymbol{B}$;
5. $\boldsymbol{b} = \gamma_1 e\tau \boldsymbol{S} + \gamma_2 \tau \boldsymbol{\Sigma} + \gamma_3 \tau \boldsymbol{Z} + \gamma_4 \tau \boldsymbol{B}$.

In Figure 1 we have plotted the horizontal stress $\overline{u_\theta u_\phi}$ versus θ for each of these cases, and for $\tau_b \Omega_0 = 1.5$, where the buoyancy timescale is defined as

$$\tau_b = (g\alpha\beta)^{-1/2} = [(g/H_p)(\nabla - \nabla_{ad})]^{-1/2} . \qquad (49a)$$

As already discussed in § 6, in case 1, corresponding to the Boussinesq approximation, the sign of $\overline{u_\theta u_\phi}$ is incorrect. In case 2, the addition of $\boldsymbol{\Sigma}$ does not change the behavior of $\overline{u_\theta u_\phi}$, as the contribution from the $\boldsymbol{\Sigma}$ term is much smaller than that of the $\boldsymbol{S}$ term. In case 3, the further addition of the vorticity $\boldsymbol{Z}$, which embodies the Λ-effect, substantially reduces the amplitude from case 1, but $\overline{u_\theta u_\phi}$ continues to have the incorrect sign. In case 4, the inclusion of the buoyancy tensor $\boldsymbol{B}$, but not the vorticity $\boldsymbol{Z}$, has a sizable numerical effect but in the wrong direction. Finally, in case 5, corresponding to the complete model, the inclusion of vorticity and buoyancy radically changes the behavior of $\overline{u_\theta u_\phi}$, which now has a sign and latitudinal dependence in good accord with observational data (Virtanen 1989) and numerical simulations (Pulkkinen et al. 1993). Our results are also in good agreement with the theoretical model of Tuominen & Rüdiger (1989), who obtained a minimum between 30°–35° S. It is important to emphasize that the agreement with observations and numerical simulations has not been achieved by adjusting parameters (we have kept the values of the constants entering our model fixed at the values determined by other independent studies, as given in Appendix D), but rather by the successive inclusion of funda-

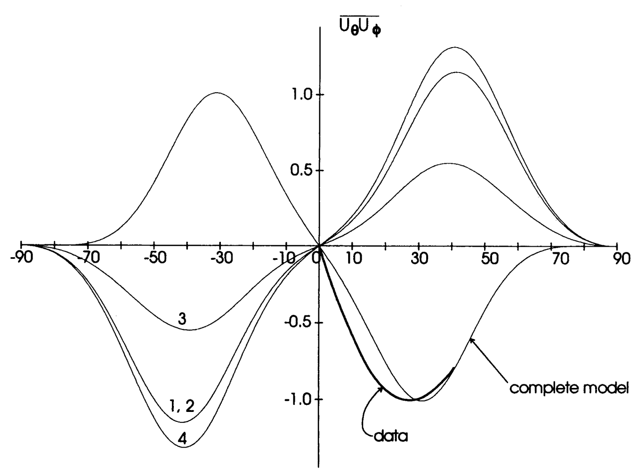

FIG. 1.—Horizontal stress $\overline{u_\theta u_\phi}$ vs. latitude in degrees (negative in the northern hemisphere and positive in the southern hemisphere) resulting from the five expressions for b_{ij} as described in § 8.1. Each curve is normalized to the maximum value of $\overline{u_\theta u_\phi}$ corresponding to case 5. The observational data are from Virtanen (1989) and Pulkkinen et al. (1993); $\tau_b \Omega_0 = 1.5$.

mental physical processses inside a star: shear, vorticity, and buoyancy.

However, *none of these effects by themselves yield a* $\overline{u_\theta u_\phi}$ *with both the correct sign and latitudinal dependence; it is only the interplay between buoyancy and vorticity that yields the correct behavior.*

8.2. *Temperature Gradient*

The value of the temperature gradient that enters equation (38a) should in principle be determined via the flux conservation equation the radial component of which, neglecting for the moment the contribution of the turbulent kinetic energy flux (Canuto 1992), is given by

$$-\chi \frac{\partial T}{\partial r} + \overline{u_r \theta} = F_T , \quad (49b)$$

where F_T is the (known) total flux, in conjunction with a stellar structure model to provide the values of χ as well as the values of the variables at the boundaries. Introducing the adiabatic temperature gradient defined in equation (3c), equation (49b) can be rewritten as

$$\beta = \beta_0 - \chi^{-1}\overline{u_r \theta}(\beta) , \quad (49c)$$

where

$$\beta = (T/H_p)(\nabla - \nabla_{ad}) , \quad \beta_0 = (T/H_p)(\nabla_r - \nabla_{ad}) . \quad (49d)$$

Clearly, since the two terms on the right-hand side of equation (49c) are similar in magnitude, that being the reason for the smallness of β, and since equation (49c) is an implicit relation for β due to the dependence of the convective flux on β itself, the solution of equation (49c) requires a delicate iterative procedure, a process well known in stellar structure calculations. The final result would yield the value of β versus θ.

In this first paper we follow a simpler procedure: we choose a set of values of τ_b so as to reproduce the available observational data and checked the consistency of assuming a θ-independent τ_b a posteriori. To exhibit the sensitivity of our results to the value of τ_b, in Figure 2*a* we plot $\overline{u_\theta u_\phi}$ for $\tau_b \Omega_0 = 1$ and 1.5 normalized to their maximum absolute value. Clearly, the observational data are consistent with a very narrow range of values of $\tau_b \Omega_0$. Horizontal stresses $\overline{u_\theta u_\phi}$ computed using values of $\tau_b \Omega_0$ larger than 1.5 have the undesirable feature of changing sign near the poles, although we note that current observational data are not very reliable near the poles. In Figure 2*b* we exhibit $\tau_b \Omega_0$ with τ_b computed via equation (49a) with the values of ∇, ∇_{ad}, g, and H_p taken from a solar model. As one can see, within the convective zone, $\tau_b \Omega_0$ is indeed of the same magnitude as that in Figure 2*a*. It may further be noticed that our values of τ_b are also consistent with those obtained by Durney (1991); specifically, his variable $\alpha = 4\Omega_0^2 \tau_b^2$ varies between 4 and 5.5 between the equator and the poles (his Fig. 2*c*).

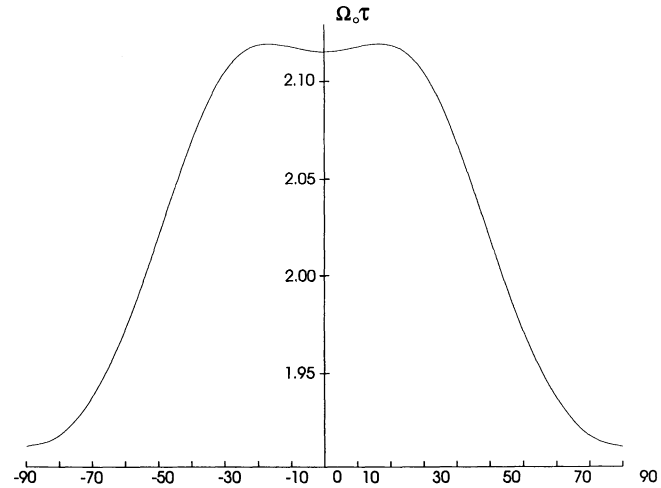

FIG. 3.—Dimensionless turbulence timescale $\tau\Omega_0$ vs. latitude for $\tau_b \Omega_0 = 1.5$.

8.3. *Turbulent Timescale $\tau(\theta)$, Dissipation Rate $\epsilon(\theta)$, and Kinetic Energy $e(\theta)$*

In Figure 3 we present $\Omega_0 \tau(\theta)$ versus θ for $\tau_b \Omega_0 = 1.5$; $\Omega_0 \tau$ is minimum at the poles, has its maximum values at $-20°$ N and 20° S, and is relatively flat near the equator. We note that the implicit τ-dependence in our model is highly nonlinear, and it

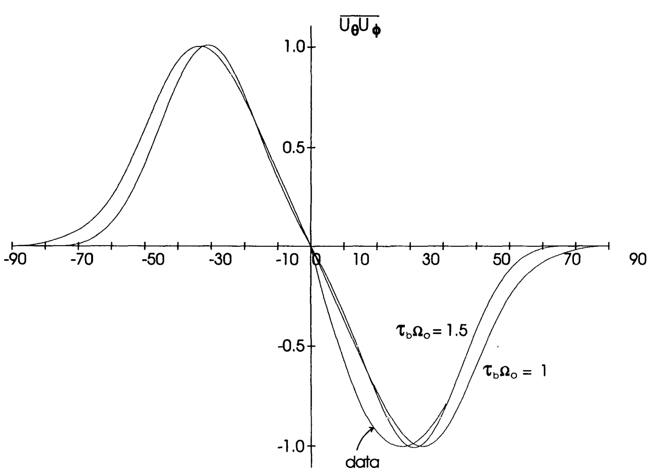

FIG. 2*a*

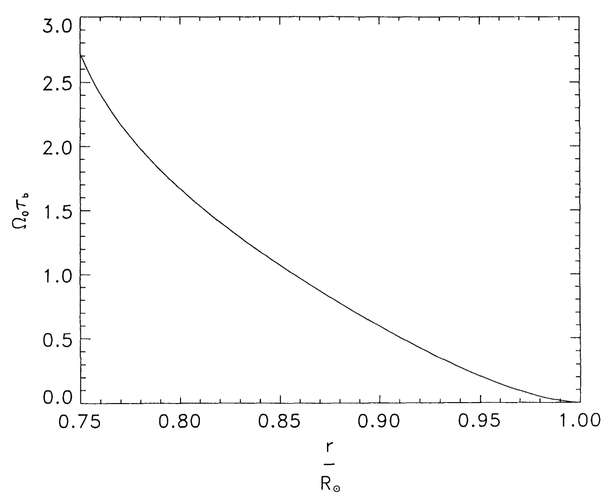

FIG. 2*b*

FIG. 2.—(*a*) Horizontal stress $\overline{u_\theta u_\phi}$ vs. latitude for $\tau_b \Omega_0 = 1$, 1.5, normalized to their maximum values. Same data as in Fig. 1. (*b*) Value of $\tau_b \Omega_0$ vs. $r/R_\odot$ as from a solar structure model.

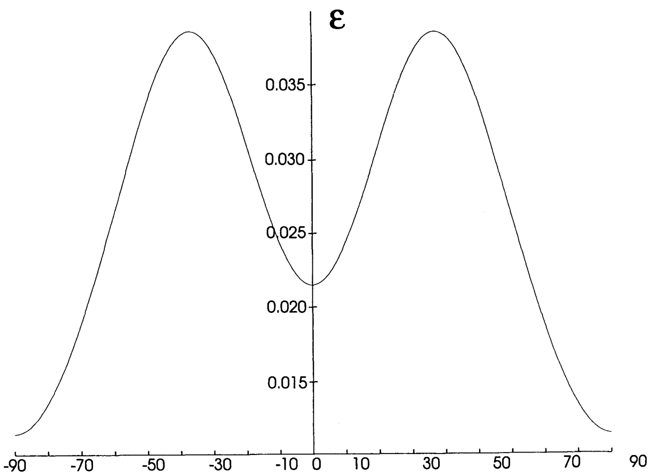

FIG. 4.—TKE dissipation rate ϵ in units of ϵ_* (eq. [45b]) vs. latitude for $\tau_b \Omega_0 = 1.5$.

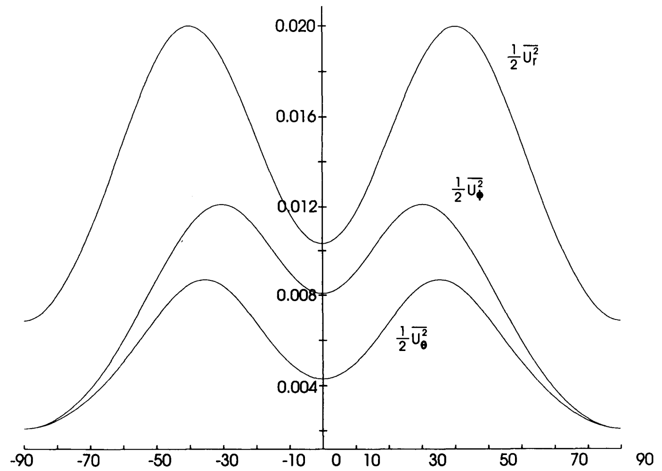

FIG. 6.—Components of the kinetic energy $\frac{1}{2}\overline{u_r^2}$, $\frac{1}{2}\overline{u_\theta^2}$, $\frac{1}{2}\overline{u_\phi^2}$ in units of e_* vs. latitude for $\tau_b \Omega_0 = 1.5$.

is difficult to assess its influence on the stresses and fluxes a priori.

In Figure 4 we present the turbulent kinetic energy dissipation rate $\epsilon(\theta)$ versus θ for $\tau_b \Omega_0 = 1.5$ obtained from the solutions of equations (44c)–(44d) as discussed in § 7. It is minimum at the poles and is maximum in the midlatitudes at $-40°$ N and 40° S. The equator is also a relative minimum of $\epsilon(\theta)$, being approximately a factor of 2 smaller than its maximum value.

In Figure 5 we present the turbulent kinetic energy $e(\theta) = \epsilon(\theta)\tau(\theta)/2$ versus θ for $\tau_b \Omega_0 = 1.5$, which is qualitatively similar to $\epsilon(\theta)$. In Figure 6 we present the three components of the turbulent kinetic energy

$$\frac{1}{2}\overline{u_r^2}\,, \quad \frac{1}{2}\overline{u_\theta^2}\,, \quad \frac{1}{2}\overline{u_\phi^2} \tag{50}$$

versus θ for $\tau_b \Omega_0 = 1.5$, which quantify the latitudinal dependence of the degree of anisotropy. All three components exhibit similar qualitative behavior: they all have their minimum values at the poles, and local minima at the equator, and their maximum values in the midlatitudes at $\sim 40°$. The equatorial minimum of the dominant component, the radial kinetic energy $\overline{u_r^2}/2$, is approximately a factor of 2 smaller than its maximum value. It is evident that the radial component dominates over each of the angular components. Clearly, the horizontal components $\overline{u_\theta^2}/2$ and $\overline{u_\phi^2}/2$ are nearly equal beyond approximately 60° S. The maximum degree of anisotropy exists in the midlatitudes, while $\overline{u_r^2}/2$ is uniformly larger than both $\overline{u_\theta^2}/2$ and $\overline{u_\phi^2}/2$, and are nearly equal beyond approximately 60° S.

8.4. *Off-Diagonal Reynolds Stresses* $\overline{u_i u_j}$

In Figure 7 we present the three off-diagonal stresses

$$\overline{u_\theta u_\phi}\,, \quad \overline{u_r u_\theta}\,, \quad \overline{u_r u_\phi} \tag{51}$$

versus θ for $\tau_b \Omega_0 = 1.5$. The amplitude of $\overline{u_r u_\theta}$ at its extreme value (45° S) and that of $\overline{u_r u_\phi}$ at its extreme value (35° S) are approximately twice and 5 times as large, respectively, as the amplitude of $\overline{u_\theta u_\phi}$ at its extreme value (30° S). Both $\overline{u_\theta u_\phi}$ and $\overline{u_r u_\theta}$ are antisymmetric, while $\overline{u_r u_\phi}$ is symmetric with respect to the equator. The stress $\overline{u_r u_\phi}$ is negative everywhere, and is in qualitative agreement with that computed at mid-layer in the simulation of Pulkkinen et al. (1993), and with that given by the model of Tuominen & Rüdiger (1989).

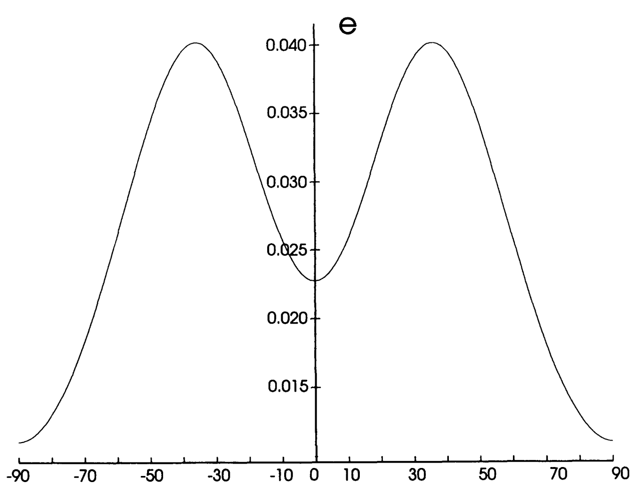

FIG. 5.—TKE e in units of e_* (eq. [45d]) vs. latitude for $\tau_b \Omega_0 = 1.5$

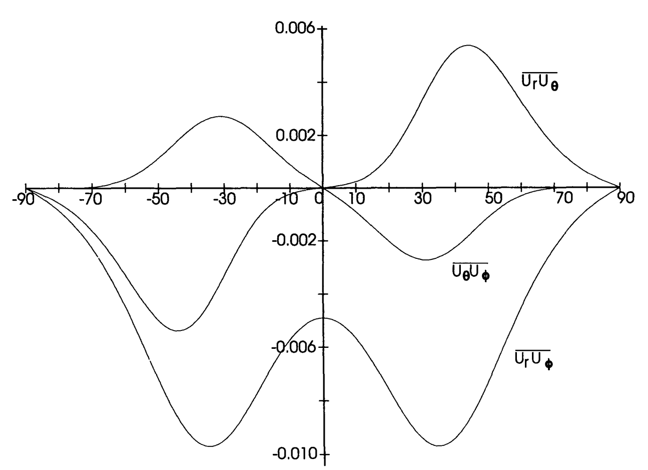

FIG. 7.—Off-diagonal Reynolds stresses $\overline{u_\theta u_\phi}$, $\overline{u_r u_\theta}$, $\overline{u_r u_\phi}$ in units of e_* vs. latitude for $\tau_b \Omega_0 = 1.5$.

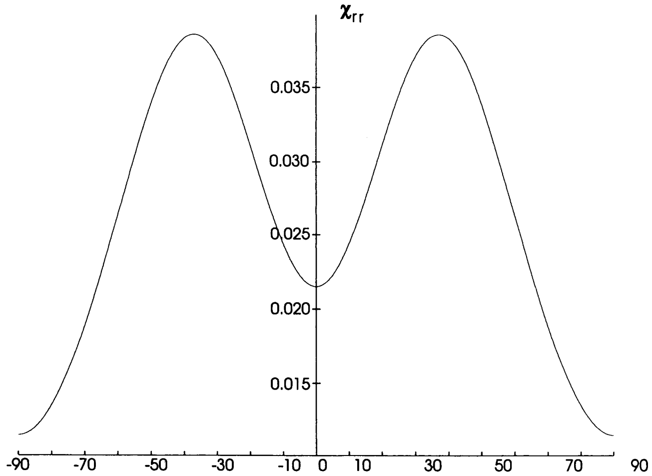

FIG. 8.—Conductivity χ_{rr}, eq. (52) in units of e_*/Ω_0 vs. latitude for $\tau_b \Omega_0 = 1.5$.

8.5. *Convective Fluxes* $\overline{u_i \theta}$

To ease the comparison with previous work, we shall write the turbulent heat fluxes as

$$\overline{u_i \theta} = \chi_{ij} \beta_j \, , \tag{52}$$

where χ_{ij} is the turbulent conductivity tensor. Since we assume that β_θ and β_ϕ are zero, we can only determine χ_{rr}, $\chi_{\phi r}$, and $\chi_{\theta r}$ while the remaining χ_{ij} remain undetermined. In Figures 8–10 we plot these conductivities versus θ for $\tau_b \Omega_0 = 1.5$: χ_{rr} and $\chi_{\phi r}$ are symmetric, while $\chi_{\theta r}$ is antisymmetric. Both $\chi_{\theta r}$ and $\chi_{\phi r}$ are approximately an order of magnitude smaller than χ_{rr}. The former have peak values around $-40°$ N and 40° S while the latter has a minimum at the poles and a relative minimum at the equator which is approximately a factor of 2 smaller than the maximum value. The conductivity χ_{rr} is positive everywhere as expected, since the convective flux and superadiabatic gradient are directed toward the surface in the convection zone. The maximum of $\chi_{\theta r}$ is at 45° S; the minimum of $\chi_{\phi r}$ is at 35°, and $\chi_{\phi r}$ is negative everywhere. Both $\chi_{\phi r}$ and $\chi_{\theta r}$ vanish at the poles, and $\chi_{\theta r}$ at the equator. From the angular dependence of $\chi_{\theta r}$, it is clear that the component $\overline{u_\theta \theta}$ of the convective flux is mainly directed away from the equator and toward the poles; in the case of $\chi_{\phi r}$, $\overline{u_\phi \theta}$ is negative everywhere. Both χ_{rr} and $\chi_{\theta r}$ are in general accord with the simulation results of Pulkkinen et al. (1993) computed just below the top of the convection zone.

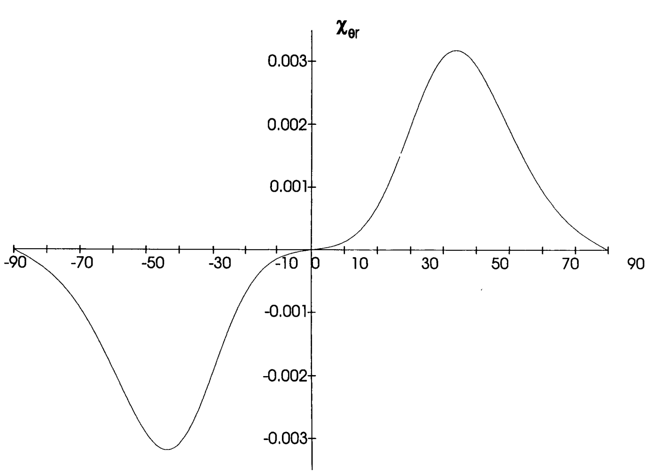

FIG. 9.—Conductivity $\chi_{\theta r}$ vs. latitude. See Fig. 8.

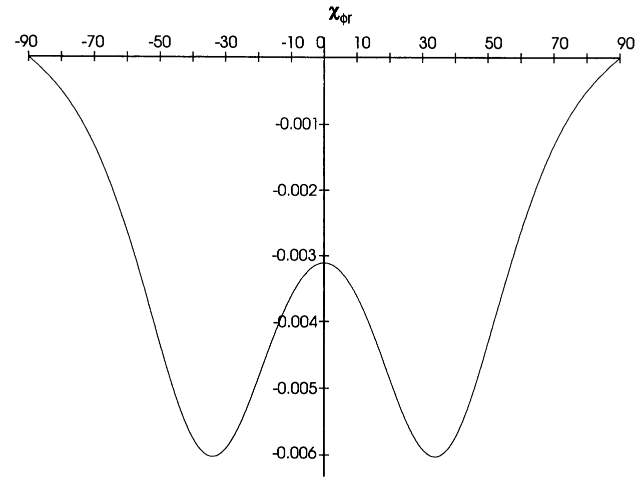

FIG. 10.—Conductivity $\chi_{\phi r}$ vs. latitude. See Fig. 8.

8.6. *Turbulent Viscosity* $\nu_t(\theta)$ *and Mixing-Length* $l(\theta)$

In Figure 11 we plot the turbulent viscosity $\nu_t(\theta)$, equation (44b), versus θ for $\tau_b \Omega_0 = 1.5$. It is minimum at the poles and is maximum in the midlatitudes at $-35°$ N and 35° S. The equator is also a relative minimum of $\nu_t(\theta)$, being nearly a factor of 2 smaller than its maximum value.

In Figure 12 we plot the mixing-length $l(\theta)$, equation (46b), versus θ for $\tau_b \Omega_0 = 1.5$. It is minimum at the poles and is maximum in the midlatitudes at $-35°$ N and 35° S. The equator is also a relative minimum of $l(\theta)$, being approximately a factor of 1.3 smaller than its maximum value.

8.7. *Where Does Solar Differential Rotation Originate?*

In Figure 13 we have plotted the ratio of the shear production P_s, equation (42a), to the total production $P = P_b + P_s$ versus θ for $\tau_b \Omega_0 = 1.5$. Since $P_s(\theta)/P(\theta)$ is negative everywhere, this *implies that shear is not a source but a sink of energy. As* $|P_s(\theta)|/P(\theta)$ *is at most* 0.025, *it means that about* 2.5% *of the*

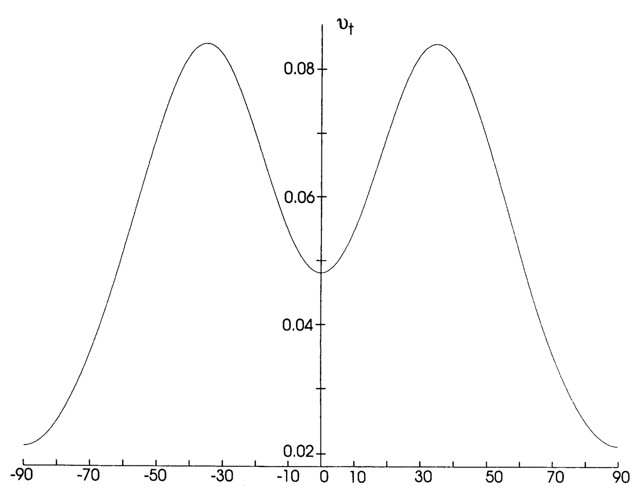

FIG. 11.—Turbulent viscosity ν_t in units of ν_t^* (eq. [46a]) vs. latitude for $\tau_b \Omega_0 = 1.5$.

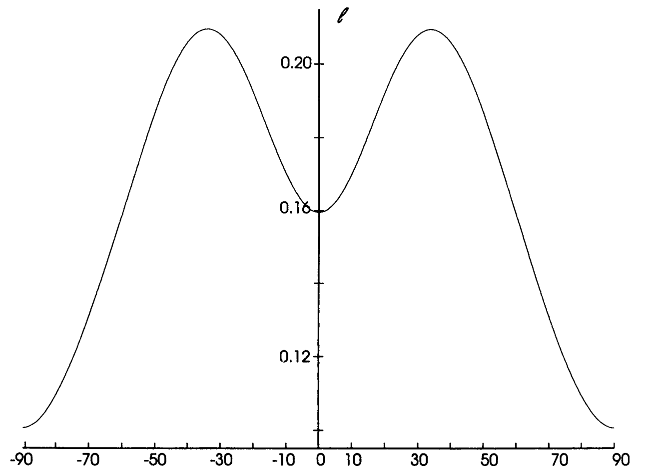

FIG. 12.—Mixing-length l (eq. [46b]) in units of l_* (eq. 46c]) vs. latitude for $\tau_b \Omega_0 = 1.5$.

total power generated by buoyancy goes to generate differential rotation. Furthermore, since the *buoyancy production* P_b/P (Fig. 14) *has a very sharp maximum around* 40°, *our model predicts that this is the latitude at which solar differential rotation originates* (note that the shear $S_{\theta\phi}$ which enters in the shear production is maximum between 50°–60°, so that the maximum of the shear production $|P_s(\theta)|/P(\theta)$ does not coincide with the maximum of the shear).

Our results show that buoyancy, which is responsible for the creation of turbulence, generates and maintains the differential rotation in the Sun, and that the source of differential rotation is in the midlatitudes. The maximum buoyancy production at about 40° coincides with the maximum of the radial component of the convective flux (which is the dominant flux), $\overline{u_r \theta} = \chi_{rr} \beta_r$, which is evident from Figure 8 since β_r is essentially independent of θ in our model as we have verified a posteriori. The turbulent kinetic energy (see Fig. 5) is also maximum at about 40°, part of which goes into generating differential rotation and part of which is dissipated into heat. A manifestation of differential rotation is the horizontal Reynolds stress $\overline{u_\theta u_\phi}$ which has its extreme values in the midlatitudes, though at angles slightly smaller than where the

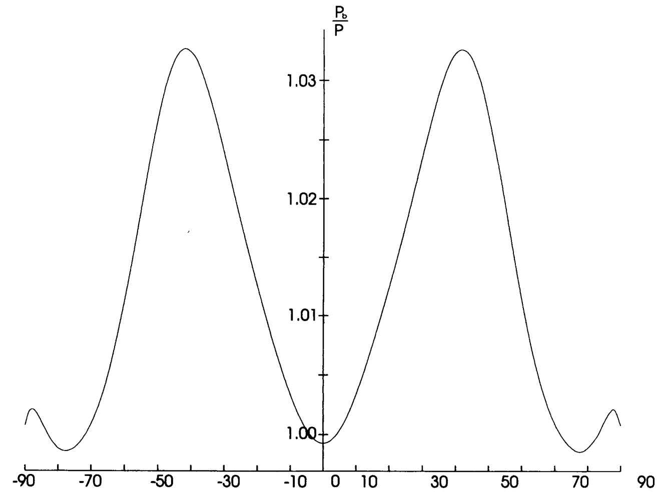

FIG. 14.—Ratio of the buoyancy production P_b (eq. [40c]) to the total production $P = P_s + P_b$ vs. latitude for $\tau_b \Omega_0 = 1.5$. Notice the maxima at around 40°.

buoyancy production and the turbulent kinetic energy are maximum.

8.8. *Comparison to Solar Data*

In Figure 15 we plot the horizontal Reynolds stress $\overline{u_\theta u_\phi}$ versus θ for $\tau_b \Omega_0 = 1.5$ in units of (deg day^{-1})^2 together with observational data pertaining to sunspot groups (Gilman & Howard 1984). In order to express our results in the same units as the observational data, we have chosen the value of $C_\mu = 0.1$ appearing in the unit ϵ_*, which is well within the range of values suggested by a variety of turbulent flows. *Both the latitudinal dependence and the amplitude of the stress* $\overline{u_\theta u_\phi}$ *predicted by our model are in very good quantitative agreement with the data from spot groups* (Gilman & Howard 1984; Gilman & Miller 1986; Virtanen 1991; Rüdiger 1989). This agreement has been achieved with one free parameter, τ_b, whose value agrees with that computed from a solar model.

9. THE Λ-EFFECT: ANISOTROPIC VISCOSITY

As discussed by Rüdiger (1989), the dependence of the Reynolds stresses on Ω itself, rather than on its derivatives $\partial\Omega/\partial r$

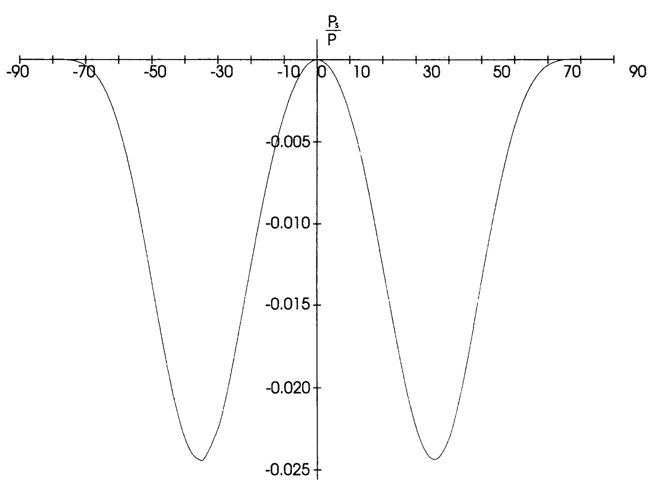

FIG. 13.—Ratio of the shear production P_s (eq. [40b]) to the total production $P = P_s + P_b$ vs. latitude for $\tau_b \Omega_0 = 1.5$.

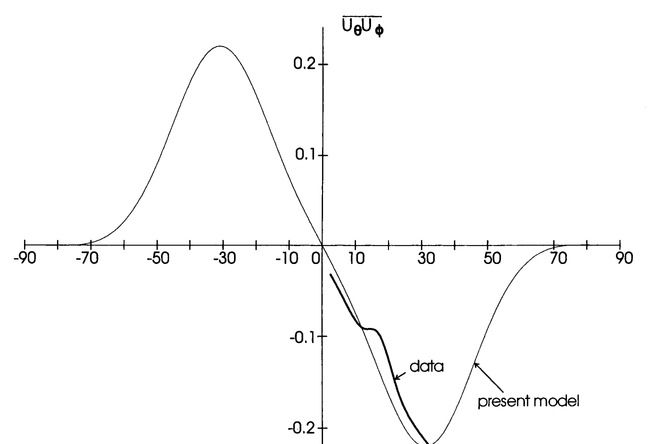

FIG. 15.—The horizontal stress $\overline{u_\theta u_\phi}$ (deg day^{-1})^2 vs. latitude for $\tau_b \Omega_0 = 1.5$. The sunspot group data are shown for comparison (Gilman & Howard 1984).

and/or $\partial\Omega/\partial\theta$, is called the Λ-effect, a phenomenon that was independently discovered more than once over the years. In our formalism, it comes out on very general grounds from the structure of the Reynolds stress, equation (27a). In fact, such an equation is a general expression for b_{ij} in terms of basic independent tensors S_{ij}, B_{ij}, Σ_{ij}, and Z_{ij}. The latter implies the vorticity tensor R_{ij} (eq. [3b]), which is the only one that entails a linear dependence on Ω (eqs. [41]). To make the Ω dependence explicit, let us consider the Reynolds stress $\overline{u_\theta u_\phi}$. Using equations (37) and (41) we obtain

$$\overline{u_\theta u_\phi} = F_1(\theta)\Omega + F_2(\theta) \sin\theta \frac{\partial\Omega}{\partial\theta} + F_3(\theta) \sin\theta r \frac{\partial\Omega}{\partial r}, \quad (53a)$$

where

$$-\frac{A}{\tau e} F_1 = 2(1-\alpha_2)\{\cos\theta[e^{-1}(\overline{u_r^2} + 2\overline{u_\theta^2}) - 2] + e^{-1}\overline{u_r u_\theta} \sin\theta\}, \quad (53b)$$

$$-\frac{A}{\tau e} F_2(\theta) = \frac{8}{15} - (1-\alpha_1)\left(e^{-1}\overline{u_r^2} - \frac{2}{3}\right) + (1-\alpha_2)[e^{-1}(\overline{u_r^2} + 2\overline{u_\theta^2}) - 2], \quad (53c)$$

$$-\frac{A}{\tau e} F_3 = (2 - \alpha_1 - \alpha_2) e^{-1}\overline{u_r u_\theta}. \quad (53d)$$

Using equations (45) and $C_\mu = 0.1$, the unit $\tau_* e_*$ takes the value 2.8×10^{15} cm^2 s^{-1}. The parameter A is defined in equation (27b). Alternatively, the variable τe can be expressed in terms of ν_t since from equation (44b)

$$\nu_t = \tfrac{1}{2} C_\mu e\tau. \quad (53e)$$

Analogously, using equations (37), (38), and (41), we derive

$$\overline{u_r u_\phi} = G_1(\theta)\Omega + G_2(\theta) \sin\theta \frac{\partial\Omega}{\partial\theta} + G_3(\theta) \sin\theta r \frac{\partial\Omega}{\partial r}, \quad (54a)$$

with

$$-\frac{A'}{\tau e} G_1 = 2(1-\alpha_2)\{[e^{-1}(2\overline{u_r^2} + \overline{u_\theta^2}) - 2] \sin\theta + e^{-1}\overline{u_r u_\theta} \cos\theta\}, \quad (54b)$$

$$-\frac{A'}{\tau e} G_2 = \left[2 - \alpha_1 - \alpha_2 - \frac{5}{2}\beta_5\left(\frac{\tau}{\tau_b}\right)^2 (1-\mu)D^{-1}\right] e^{-1}\overline{u_r u_\theta}, \quad (54c)$$

$$-\frac{A'}{\tau e} G_3 = \frac{8}{15} - (1-\alpha_1)\left(e^{-1}\overline{u_\theta^2} - \frac{2}{3}\right) + (1-\alpha_2) \times [e^{-1}(2\overline{u_r^2} + \overline{u_\theta^2}) - 2] - \frac{5}{2}\beta_5\left(\frac{\tau}{\tau_b}\right)^2 D^{-1} e^{-1}\overline{u_r^2}, \quad (54d)$$

where

$$A' = A + \frac{25}{4}\beta_5\left(\frac{\tau}{\tau_b}\right)^2 (1-\mu)D^{-1}(f_1 + B),$$

$$\mu \equiv (1-\gamma_1)(f_1 + B)^{-1} C_*^{-1}\left(\frac{\tau}{\tau_b}\right)^2,$$

$$D \equiv [\omega_r^2 + (1-\mu)\omega_\theta^2] \sin^2\theta - \frac{25}{4}(1-\mu)(f_1 + B)^2,$$

$$\omega_r \equiv \tau r \frac{\partial\Omega}{\partial r}, \quad \omega_\theta \equiv \tau \frac{\partial\Omega}{\partial\theta}. \quad (54e)$$

These expressions can now be compared with the general formulae in terms of four independent turbulent viscosities (Rüdiger 1989):

$$\nu_{hh}, \quad \nu_{hv}, \quad \nu_{vh}, \quad \nu_{vv}, \quad (55a)$$

i.e.,

$$\overline{u_\theta u_\phi} = \Lambda_H \cos\theta\Omega - \nu_{hh} \sin\theta \frac{\partial\Omega}{\partial\theta} - \nu_{hv} \cos\theta r \frac{\partial\Omega}{\partial r}, \quad (55b)$$

$$\overline{u_r u_\phi} = \Lambda_V \sin\theta\Omega - \nu_{vh} \cos \frac{\partial\Omega}{\partial\theta} - \nu_{vv} \sin\theta r \frac{\partial\Omega}{\partial r}. \quad (55c)$$

We thus have the following identifications:

$$\Lambda_H \cos\theta = F_1, \quad \Lambda_V \sin\theta = G_1$$
$$\nu_{hh} = -F_2, \quad \nu_{hv} = -F_3 \tan\theta \quad (55d)$$
$$\nu_{vh} = -G_2 \tan\theta, \quad \nu_{vv} = -G_3$$

Using an extension of the mixing length, Durney (1991) has computed the four viscosities which he calls

$$\nu_1 \equiv \nu_{vv}, \quad \nu_2 \tan\theta \equiv \nu_{vh}, \quad \nu_3 \tan\theta \equiv \nu_{hv}, \quad \nu_4 \equiv \nu_{hh}. \quad (55e)$$

Küker, Rüdiger, & Kichatinov (1993) have used equations (55b)–(55c) under the approximation

$$\nu_{hv} = \nu_{vh} = (\nu_{\rm I} - \nu_{\rm II}) \sin^2\theta, \quad (55f)$$

and with the transformation

$$\nu_{hh} = \nu_{\rm II} \sin^2\theta + \nu_{\rm I} \cos^2\theta,$$
$$\nu_{vv} = \nu_{\rm II} \cos^2\theta + \nu_{\rm I} \sin^2\theta. \quad (55g)$$

Other authors have used much simplified expressions. Zahn (1992) uses equation (55b)–(55c) in the form

$$\overline{u_\theta u_\phi} = -\nu_{hh} \sin\theta \frac{\partial\Omega}{\partial\theta},$$
$$\overline{u_r u_\phi} = -\nu_{vv} \sin\theta r \frac{\partial\Omega}{\partial r}. \quad (55h)$$

This is just the Boussinesq formula (32a) whose inadequacy we have already discussed in § 6.

As for the present model, we have for ν_{hv} and ν_{vh},

$$\frac{\nu_{vh}}{\nu_{hv}} = \frac{A}{A'}\left[1 - C\left(\frac{\tau}{\tau_b}\right)^2 (1-\mu)D^{-1}\right], \quad (56a)$$

where $2C = 5\beta_5(2 - \alpha_1 - \alpha_2)^{-1}$. Because of *buoyancy*

$$\nu_{vh} \neq \nu_{hv}. \quad (56b)$$

The function ν_{vh}/ν_{hv} is plotted in Figure 16*a*. Next, since if the viscosity in the radial direction is less than that in the horizontal direction, the angular velocity increases toward the equator, the variable

$$s = \frac{\nu_{hh}}{\nu_{vv}} + \frac{1}{2}\frac{\Lambda_H}{\nu_{vv}} \quad (56c)$$

must be greater than unity. Köhler (1969) has suggested the following empirical rule:

$$\frac{\Omega(\text{equator})}{\Omega(\text{pole})} = 1 + \frac{11}{10}(s - 1). \quad (56d)$$

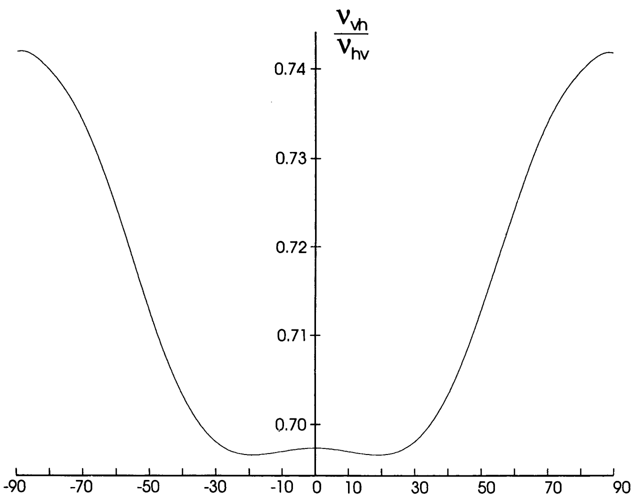

FIG. 16*a*

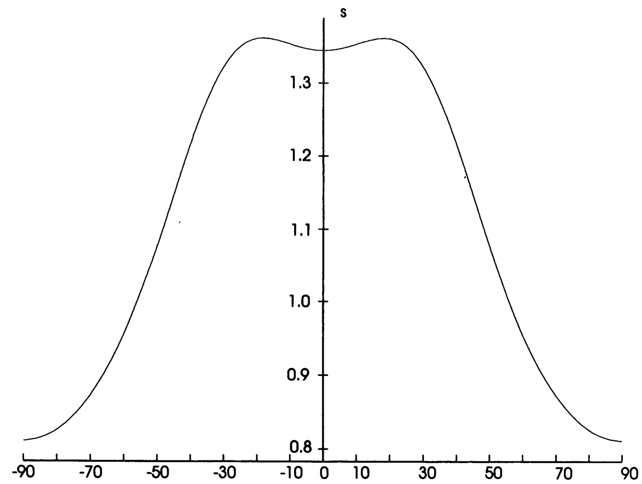

FIG. 16*b*

FIG. 16.—(*a*) Ratio v_{vh}/v_{hv} (eq. [56a]) vs. latitude. (*b*) Parameter s (eq. [56c]) vs. latitude

Using equation (43b), the left-hand side of equation (56d) is 1.41, which implies that

$$s = 1.371 \ . \tag{56e}$$

Since we have assumed that the angular velocity is larger at the equator, for consistency reasons, our model must reproduce equation (56e). In Figure 16*b* we plot the variable s predicted by our model. The values are indeed consistent with equation (56e).

10. FRAME INDEPENDENCE

It may be of interest to study the effect of writing the basic equations in a noninertial frame rotating with a constant angular velocity Ω_0. Equation (A1) would acquire the extra term corresponding to a Coriolis force

$$-2\epsilon_{ijk}\Omega_{0j} \ , \tag{57a}$$

where ϵ_{ijk} is the totally antisymmetric unit tensor. Correspondingly, equation (A23) acquires the new term

$$-2(\epsilon_{ilk}\overline{u_j u_k} + \epsilon_{jlk}\overline{u_i u_k})\Omega_{0l} \ , \tag{57b}$$

which can be reabsorbed in the first two terms of equation (A23) which change to

$$-\left[\overline{u_j u_k}\left(\frac{\partial U'_i}{\partial x_k} + 2\epsilon_{ilk}\Omega_{0l}\right) + \overline{u_i u_k}\left(\frac{\partial U'_j}{\partial x_k} + 2\epsilon_{jlk}\Omega_{0l}\right)\right] , \tag{57c}$$

where the prime on U' indicates that the mean velocity is now taken in the rotating system. Introducing now the shear S'_{ij}, equation (3a) and the *modified vorticity*

$$R^*_{ij} = \frac{1}{2}\left(\frac{\partial U'_i}{\partial x_j} - \frac{\partial U'_j}{\partial x_i}\right) - 2\epsilon_{ijk}\Omega_{0k} \ . \tag{57d}$$

Equation (57c) becomes

$$-[\overline{u_j u_k}(S'_{ik} + R^*_{ik}) + \overline{u_i u_k}(S'_{jk} + R^*_{jk})] \ , \tag{57e}$$

which is formally identical to the expression in the absence of rotation, the difference being the change of S_{ij} and R_{ij} into S'_{ij} and R^*_{ij}. Clearly, since S'_{ij} depends only on the derivatives of Ω, the fact that the prime means that one should change Ω into $\Omega - \Omega_0$ has no effect, and thus, $S'_{ij} = S_{ij}$. In the case of the vorticity we have, using equation (41c),

$$\begin{aligned} R^*_{\theta\phi} &= -2(\Omega - \Omega_0)\cos\theta - S_{\theta\phi} - 2\epsilon_{231}\Omega_{01} \\ &= -2(\Omega - \Omega_0)\cos\theta - S_{\theta\phi} - 2\Omega_0\cos\theta \ , \end{aligned} \tag{57f}$$

which shows explicitly that Ω_0 cancels out. This means that

$$R^*_{ij} \to R_{ij} + \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) , \tag{57g}$$

and thus the effect of a constant Ω_0 cancels out entirely from the equations.

11. PLANS FOR FUTURE RESEARCH

The Reynolds stress model solved in this paper is algebraic, rather than differential, so that boundary and initial conditions could not be imposed. For example, the stresses $\overline{u_r u_\theta}$ and $\overline{u_r u_\phi}$, which ought to vanish at the surface, do not do so in our case. It is however important to note that both stresses are in good accord with the stresses *computed below the surface of the convection zone* in the simulations of Durney (1991) and Pulkkinen et al. (1993). This indicates that our results ought to be considered valid near the surface, but not necessarily exactly at the surface.

The nature of the formalism we have presented has the advantage of allowing us to envisage a series of improvements upon the model solved in this paper. For example,

1. The nonlinear contributions to the Reynolds stresses represented by Π_{ij}(NL) and Π^θ_i(NL) ought to be included.
2. The M5 model could be supplemented with the *differential equations for the turbulent kinetic energy e.*
3. While keeping $\overline{\theta^2}$ in algebraic form, one could solve the differential equations for the Reynolds stresses with a simple down-gradient approximation for the third-order moments.
4. The next step involves the improved treatment of the diffusion terms discussed in Appendix C.
5. All previous cases could be implemented with a rotation curve for Ω either given by the data, as we have done, or as a result of solving the angular momentum equation.
6. Any of the previous models could be solved in conjunction with a stellar evolution model that would imply the solution of the flux equation as discussed in § 8.2.

12. CONCLUSIONS

Since the angular momentum transport in a differentially rotating fluid is contributed principally by the Reynolds stresses, the study of the latter has been a topic of major importance. The most widely used models to construct $\overline{u_i u_j}$ are still phenomenological in nature, using symmetry considerations and general physical constraints (see the extensive monograph by Rüdiger 1989), or extensions of the standard mixing-length theory to rotating convection (Kichatinov 1986, 1987; Durney 1987, 1991; Küker et al. 1993). Phenomenological approaches express the stresses and fluxes in relatively simple forms that exhibit the proper *qualitative* physics, but they also contain a large number of undetermined parameters that severely limit their predictive power. In addition, because of their nature, there is no systematic methodology to improve upon them.

In the last decade, numerical simulations of the basic equations have acquired an increasingly important role in both stellar structure calculations (Nordlund 1982; Sofia & Chan 1984; Marcus 1986; Chan & Sofia 1986, 1989; Stein & Nordlund 1989; Stein, Nordlund, & Kuhn 1989; Hossain & Mullan 1991; Xie & Toomre 1991, 1993) as well as in the problems of differential rotation (Glatzmaier 1984, 1985a, b, 1987; Gilman & Miller 1986; Pulkkinen et al. 1993). Both approaches share a common difficulty: since it is not possible to resolve all of the scales characterizing a turbulent flow with a Reynolds number of Re $\sim 10^{14}$ (we recall that the total number of grid points scales as $\sim \mathrm{Re}^3$), one must rely on a *Large-Eddy Simulation* (*LES*) *which is only as good as the input physical model to describe the scales that cannot be numerically resolved.* Since a subgrid scale (SGS) model appropriate for stellar interiors has not yet become available, the universal use of the Smagorinsky (1963) formula as an SGS model introduces an unquantified uncertainty in all of the LES results since such a model was originally devised to describe shear flows without buoyancy and/or rotation which are key features of stellar structure.

Here we propose an intermediate approach which is not phenomenological since it derives from the basic governing dynamical equations and yet, it has an algebraic rather than numerical structure. We began with the derivation of the *dynamical* equations for all of the turbulence variables of interest, and presented a full treatment of the nonlinear interactions, which considerably improves the mixing-length approach in which all the nonlinear interaction effects are combined into a relaxation time (Kichatinov 1986; Durney 1987, 1991). This is explicitly seen in the fact that the equations for the Reynolds stresses and convective fluxes, which are second-order moments, imply third-order moments, for which we also derive the dynamic equations and suggest a systematic procedure to solve them. The latter is of greater validity than the down-gradient approximation often adopted to treat these moments (Appendix C).

Were the complete Reynolds stress model M1 to be solved, it would predict the absolute latitudinal and depth dependence of the Reynolds stresses $\overline{u_i u_j}$, convective fluxes $\overline{u_i \theta}$, and the other quantities that characterize the highly turbulent motion in the convectively unstable regions of stars. However, in view of the complexity of the full model, we have introduced a set of simplifying assumptions so as to obtain *algebraic* Reynolds stress models of varying complexity. In this paper, we have solved the model that contains only one differential equation for the dissipation rate ϵ, while all the other turbulence variables are given in algebraic form. We have presented a detailed study of the separate effects of shear, rotation, anisotropic energy production, and buoyancy on the angular behavior of $\overline{u_i u_j}$ and $\overline{u_i \theta}$ and have shown that:

1. $\overline{u_\theta u_\phi}$ is in quantitative agreement with observational data (Gilman & Howard 1984).
2. Shear alone, namely the Boussinesq formula $\overline{u_i u_j} = -\nu_t S_{ij}$, cannot give the expected result since it describes a flow in which turbulence is generated by shear while in this case shear is generated by turbulence (which derives from an independent source, buoyancy).
3. Shear and buoyancy alone do not yield acceptable results.
4. Agreement with the data is the result of the nonlinear interplay between vorticity and buoyancy.
5. Production of shear by turbulent buoyancy is predicted to occur mostly at a latitude of $\sim 40°$.
6. From the energy viewpoint, 2.5% of the buoyant production rate is required to generate and maintain solar differential rotation.
7. The degree of anisotropy in the turbulent viscosity is predicted to depend on latitude with an average difference amounting to $\sim 20\%$.

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APPENDIX A

DERIVATION OF THE BASIC EQUATIONS

The basic equations describing a compressible fluid of density $\tilde{\rho}$, pressure $\tilde{p}$, temperature $\tilde{T}$, velocity v_i, kinematic viscosity ν and thermal conductivity K, in an inertial reference frame are given by ($d/dt \equiv \partial/\partial t + v_j \partial/\partial x_j$)

$$\tilde{\rho}\,\frac{dv_i}{dt} = -\frac{\partial \tilde{p}}{\partial x_i} - g_i \tilde{\rho} + \nu \tilde{\rho}\,\frac{\partial^2}{\partial x_j^2}\,v_i\,, \tag{A1}$$

$$\tilde{\rho} c_p \frac{d\tilde{T}}{dt} = \frac{d\tilde{p}}{dt} + K\,\frac{\partial^2 \tilde{T}}{\partial x_j^2} + \mu\left[\frac{\partial^2}{\partial x_i \partial x_j}\,v_i v_j + \left(\frac{\partial v_i}{\partial x_j}\right)^2\right] + c_p \tilde{\rho} Q\,, \tag{A2}$$

where $c_p \tilde{\rho} Q$ is the gradient of an external flux, and $\mu = \tilde{\rho}\nu$. In addition, we assume a perfect gas law $\tilde{p} = R\tilde{\rho}\tilde{T}$. First, we split the variables into a mean and a fluctuating part

$$\tilde{p} = P + p\,, \qquad \tilde{T} = T + \theta\,, \qquad \tilde{\rho} = \rho + \rho'\,, \qquad v_i = U_i + u_i\,, \tag{A3}$$

where, P, T, ρ, and U_i represent the mean fields; the fluctuating components have zero average

$$\bar{p} = \bar{\theta} = \overline{\rho'} = \overline{u_i} = 0 \, . \tag{A4}$$

Assuming further that the fluid is an ideal gas, $P = R\rho T$, and neglecting second order quantities, we derive for ρ' the relation

$$\frac{\rho'}{\rho} = -\alpha\theta + \frac{p}{P} , \quad \alpha \equiv \frac{1}{T} . \tag{A5}$$

In the standard Boussinesq approximation (Canuto 1992, hereafter Paper I, eq. [21b]), the second term with the fluctuating pressure p/P is absent. The physical interpretation of the terms arising from its inclusion will be discussed shortly. Inserting equations (A3)–(A5) into equations (A1)–(A2), and following the procedure outlined in Paper I, we obtain the dynamical equations for the mean and fluctuating variables ($D/Dt \equiv \partial/\partial t + U_i \partial/\partial x_i$, $\lambda_i = g_i \alpha$; for any vector a_i, $a_{i,j} \equiv \partial a_i/\partial x_j$):

$$\frac{DU_i}{Dt} = -\left(g_i + \frac{1}{\rho}\frac{\partial P}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\overline{u_i u_j} + N_1^i \, , \tag{A6}$$

$$\frac{Du_i}{Dt} = -u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}(u_i u_j - \overline{u_i u_j}) + \lambda_i \theta + \nu \frac{\partial^2 u_i}{\partial x_j^2} + N_2^i \, , \tag{A7}$$

$$\frac{DT}{Dt} = \chi \frac{\partial^2 T}{\partial x_j^2} - \frac{\partial}{\partial x_j}\overline{u_j \theta} + \frac{\epsilon}{c_p} + \frac{1}{c_p} U_j \frac{\partial P}{\partial x_j} + Q + N_3 \, , \tag{A8}$$

$$\frac{D\theta}{Dt} = -u_j \frac{\partial T}{\partial x_j} - \frac{\partial}{\partial x_j}(u_j\theta - \overline{u_j\theta}) + \chi \frac{\partial^2 \theta}{\partial x_j^2} + \frac{\nu}{c_p}[(u_{i,j})^2 - \overline{(u_{i,j})^2}] + N_4 \, , \tag{A9}$$

where we have taken the $\nu \to 0$ limit and where ϵ is the dissipation rate of kinetic energy. The non-Boussinesq terms N's are given by (Canuto 1993, hereafter Paper II, eqs. [28]–[33])

$$\rho N_1^i \equiv -\alpha\overline{\theta \frac{\partial p}{\partial x_i}} + \overline{\frac{p}{P}\frac{\partial p}{\partial x_i}} , \tag{A10}$$

$$N_2^i \equiv -\left(g_i + \frac{1}{\rho}\frac{\partial P}{\partial x_i}\right)\alpha\rho\theta + \frac{p}{P}\frac{\partial P}{\partial x_i} - \alpha\left(\theta \frac{\partial p}{\partial x_i} - \overline{\theta \frac{\partial p}{\partial x_i}}\right) + \frac{1}{P}\left(p \frac{\partial p}{\partial x_i} - \overline{p \frac{\partial p}{\partial x_i}}\right) , \tag{A11}$$

$$\rho c_p N_3 \equiv U_j \Lambda_j + \overline{u_j \frac{\partial p}{\partial x_j}} + \left(\alpha\overline{\theta u_j} - \frac{1}{P}\overline{p u_j}\right)\frac{\partial P}{\partial x_j} + \alpha\overline{\theta u_j \frac{\partial p}{\partial x_j}} - \frac{1}{P}\overline{p u_j \frac{\partial p}{\partial x_j}} , \tag{A12}$$

with

$$\Lambda_j \equiv \alpha\overline{\theta \frac{\partial p}{\partial x_j}} - \frac{1}{P}\overline{p \frac{\partial p}{\partial x_j}} , \tag{A13}$$

$$c_p \rho N_4 \equiv U_j \Delta_j + u_j \frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j}(p u_j - \overline{p u_j}) + \left[\alpha(\theta u_i - \overline{\theta u_i}) - \frac{1}{P}(p u_i - \overline{p u_i})\right]\frac{\partial P}{\partial x_i} + \alpha\left(u_i \theta \frac{\partial p}{\partial x_i} - \overline{u_i \theta \frac{\partial p}{\partial x_i}}\right) - \frac{1}{P}\left(u_i p \frac{\partial p}{\partial x_i} - \overline{u_i p \frac{\partial p}{\partial x_i}}\right) , \tag{A14}$$

with

$$\Delta_i \equiv \frac{\partial p}{\partial x_i} - \left(\frac{p}{P} - \alpha\theta\right)\frac{\partial P}{\partial x_i} + \alpha\left(\theta \frac{\partial p}{\partial x_i} - \overline{\theta \frac{\partial p}{\partial x_i}}\right) - \frac{1}{P}\left(p \frac{\partial p}{\partial x_i} - \overline{p \frac{\partial p}{\partial x_i}}\right) . \tag{A15}$$

A1. SECOND-ORDER MOMENTS

Following the procedure outlined in Papers I and II, we derive the following results:

$$\frac{D}{Dt}\overline{u_i \theta} + \frac{\partial}{\partial x_j}\overline{\theta u_i u_j} = -\left(\overline{u_i u_j}\frac{\partial T}{\partial x_j} + \overline{u_j \theta}\frac{\partial U_i}{\partial x_j}\right) + \lambda_i \overline{\theta^2} - \Pi_i^\theta + \eta_i + C_i \, , \tag{A16}$$

where the pressure correlation term is defined as

$$\Pi_i^\theta \equiv \overline{\theta \frac{\partial p}{\partial x_i}} , \tag{A17}$$

and the new term C_i is defined by

$$C_i \equiv \overline{N_2^i \theta} + \rho \overline{N_4 u_i} + \frac{\rho}{c_n} \overline{u_i \epsilon} , \tag{A18}$$

with (Paper I, eq. [37f])

$$\overline{u_i \epsilon} = \frac{1}{\tau} \overline{q^2 u_i} . \tag{A19}$$

Next, consider the equation for the temperature variance $\overline{\theta^2}$. In lieu of equation (35a) of Paper I we have

$$\frac{D\overline{\theta^2}}{Dt} + \frac{\partial}{\partial x_i} \overline{u_i \theta^2} = -2\overline{u_i \theta} \frac{\partial T}{\partial x_i} + \chi \frac{\partial^2 \overline{\theta^2}}{\partial x_j^2} - 2\epsilon_\theta + C^\theta , \tag{A20}$$

where

$$\frac{1}{2} C^\theta \equiv \overline{N_4 \theta} + \frac{1}{c_p} \overline{\epsilon \theta} , \tag{A21}$$

with (Paper I, eq. [38e])

$$\overline{\epsilon \theta} = \frac{1}{\tau} \overline{q^2 \theta} . \tag{A22}$$

Next, consider the equation for the Reynolds stress $\overline{u_i u_j}$. We derive

$$\frac{D}{Dt} \overline{u_i u_j} + \frac{\partial}{\partial x_k} \overline{u_i u_j u_k} = -\left(\overline{u_j u_k} \frac{\partial U_i}{\partial x_k} + \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} \right) + \lambda_i \overline{u_j \theta} + \lambda_j \overline{u_i \theta} - \Pi_{ij} + \nu \frac{\partial^2}{\partial x_k^2} \overline{u_i u_j} - \epsilon_{ij} + C_{ij} , \tag{A23}$$

where we have defined

$$\Pi_{ij} \equiv \overline{u_i \frac{\partial p}{\partial x_j}} + \overline{u_j \frac{\partial p}{\partial x_i}} , \qquad C_{ij} = \overline{N_2^i u_j} + \overline{N_2^j u_i} , \tag{A24a}$$

$$\epsilon_\theta \equiv \chi \overline{\left(\frac{\partial \theta}{\partial x_j} \right)^2} , \qquad \epsilon_{ij} = 2\nu \overline{u_{i,k} u_{j,k}} . \tag{A24b}$$

Even though the tensor ϵ_{ij} is usually taken to be diagonal, we prefer to use a slightly more general expression

$$\epsilon_{ij} = \frac{2}{3} \epsilon \delta_{ij} + (1 - F^{1/2}) \frac{\epsilon}{e} b_{ij} , \tag{A25}$$

where the definition of F is given in Appendix D. The equation for the turbulent kinetic energy $e \equiv \frac{1}{2}\overline{q^2}$, where $q^2 \equiv u_i u_i$, is then,

$$\frac{De}{Dt} + \frac{\partial}{\partial x_i} \frac{1}{2} \overline{q^2 u_i} = -\overline{u_i u_j} \frac{\partial U_i}{\partial x_j} + \lambda_i \overline{u_i \theta} - \frac{1}{2} \Pi_{ii} - \epsilon + \frac{1}{2} C_{ii} . \tag{A26}$$

A2. PRESSURE-STRAIN CORRELATIONS

There have been many proposals for the tensor Π_{ij} which have recently been reviewed by Shih & Shabbir (1992), whose formulation we follow

$$\Pi_{ij} = 2c_4 \tau^{-1} b_{ij} + (1 - \beta_5) B_{ij} - \frac{4}{5} e S_{ij} - \alpha_1 \Sigma_{ij} - \alpha_2 Z_{ij} + \Pi_{ij}(\mathrm{NL}) + \frac{\partial}{\partial x_j} \overline{p u_i} + \frac{\partial}{\partial x_i} \overline{p u_j} , \tag{A27a}$$

where

$$\Pi_{ij}(\mathrm{NL}) = -e^{-1} \Pi_{ij}^{\mathrm{nl}} + \lambda_k \Delta_{ij}^k . \tag{A27b}$$

The first term is known as the Rotta (1951) term. The timescale τ, which in principle should be determined from the integral of a Green function derived from a second-order closure (Herring 1987), is usually taken to be $\tau = 2e/\epsilon$. The nonlinear contributions Π_{ij}^{nl} and Δ_{ij}^k are given by

$$5\Pi_{ij}^{\mathrm{nl}} = b_{ik}^2 S_{jk} + b_{jk}^2 S_{ik} - 2b_{kj} b_{mi} S_{km} - 3b_{ij} b_{km} S_{km} + b_{ik}^2 R_{jk} + b_{jk}^2 R_{ik} , \tag{A28}$$

$$\Delta_{ij}^k \equiv -\tfrac{2}{3}(1 + 4\beta_5) A_{ij}^k + \tfrac{2}{3}(1 - \tfrac{1}{2}\beta_5) C_{ij}^k + (\beta_7 + 3\beta_5) D_{ij}^k + \beta_9 E_{ij}^k + e^{-1}(\beta_5 - 1) b_{ij} \overline{\theta u_k} + (\tfrac{3}{2}) e^{-2} \beta_5 b_{ij} b_{kp} \overline{\theta u_p} , \tag{A29}$$

where our b_{ij} and the one in Shih & Lumley (1985) are related by: $b_{ij} = 2eb_{ij}^{\mathrm{SL}}$. Furthermore,

$$b_{ij} = \overline{u_i u_j} - \tfrac{2}{3} e \delta_{ij} , \qquad b_{ij}^2 \equiv b_{ik} b_{jk} , \tag{A30}$$

$$B_{ij} = \lambda_i \overline{u_j \theta} + \lambda_j \overline{u_i \theta} - \tfrac{2}{3} \delta_{ij} \lambda_k \overline{u_k \theta} , \tag{A31}$$

$$\Sigma_{ij} = S_{ik}b_{kj} + S_{jk}b_{ik} - \tfrac{2}{3}\delta_{ij}S_{kl}b_{kl}\,, \tag{A32}$$

$$Z_{ij} = R_{ik}b_{kj} + R_{jk}b_{ik} - \tfrac{2}{3}\delta_{ij}R_{kl}b_{kl}\,, \tag{A33}$$

$$2S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\,, \qquad 2R_{ij} = \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\,, \tag{A34}$$

$$2eA^k_{ij} \equiv b_{ik}\overline{\theta u_j} + b_{jk}\overline{\theta u_i} - \tfrac{2}{3}\delta_{ij}b_{pk}\overline{\theta u_p}\,, \tag{A35}$$

$$2eC^k_{ij} \equiv (\delta_{ik}b_{jp} + \delta_{jk}b_{ip} - \tfrac{2}{3}\delta_{ij}b_{kp})\overline{u_p\theta}\,, \tag{A36}$$

$$4e^2D^k_{ij} \equiv [b_{ik}b_{jp} + b_{jk}b_{ip} - (\delta_{ik}b_{jm} + \delta_{jk}b_{im})b_{mp}]\overline{u_p\theta}\,, \tag{A37}$$

$$4e^2E^k_{ij} \equiv (b_{im}\overline{u_j\theta} + b_{jm}\overline{u_i\theta})b_{mk} - (\delta_{ik}b_{jm} + \delta_{jk}b_{im})b_{mp}\overline{u_p\theta}\,, \tag{A38}$$

$$\Delta^k_{ii} = 0\,. \tag{A39}$$

If we neglect the nonlinear terms we recover equations (44) and (44a) of Paper I provided we call $1 - \beta_5 \equiv c_5$. Combining equations (A23), (A26), and (A27), we obtain the dynamic equation for the tensor b_{ij}

$$\frac{D}{Dt}b_{ij} + \frac{\partial}{\partial x_k}\left[\left(\overline{u_iu_ju_k} - \frac{1}{3}\delta_{ij}\overline{q^2u_k}\right) + \left(\delta_{ij}\overline{pu_j} + \delta_{jk}\overline{pu_i} - \frac{2}{3}\delta_{ij}\overline{pu_k}\right)\right]$$
$$= -2c_4^*\tau^{-1}b_{ij} + \beta_5B_{ij} - \frac{8}{15}eS_{ij} - (1-\alpha_1)\Sigma_{ij} - (1-\alpha_2)Z_{ij} - \Pi_{ij}(\mathrm{NL}) + C_{ij} - \frac{1}{3}\delta_{ij}C_{kk}\,. \tag{A40}$$

Similarly,

$$\Pi_i^\theta \equiv \overline{\theta\frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{p\theta} + f_1\tau^{-1}\overline{u_i\theta} + \gamma_1\lambda_i\overline{\theta^2} - \frac{3}{4}\alpha_3\left(S_{ij} + \frac{5}{3}R_{ij}\right)\overline{\theta u_j} + \Pi_i^\theta(\mathrm{NL}) - (\nu+\chi)\overline{\frac{\partial\theta}{\partial x_k}\frac{\partial u_i}{\partial x_k}}\,, \tag{A41a}$$

where

$$\Pi_i^\theta(\mathrm{NL}) \equiv \lambda_i Y_{ij} + e^{-1}(\alpha_4B^k_{ij} - \alpha_6b_{ij}\overline{\theta u_k})(S_{jk} + R_{jk}) \tag{A41b}$$

and

$$4eY_{ij} \equiv 2\gamma_2\overline{\theta^2}b_{ij} + 4\gamma_3\overline{\theta u_i}\,\overline{\theta u_j} + 2\gamma_4e^{-1}(b_{ik}\overline{\theta u_j}\,\overline{\theta u_k} + b_{jk}\overline{\theta u_i}\,\overline{\theta u_k}) + \gamma_5e^{-1}b_{ik}b_{kj}\overline{\theta^2} \tag{A42}$$

$$B^k_{ij} \equiv b_{ik}\overline{\theta u_j} + b_{jk}\overline{\theta u_i} - \tfrac{2}{3}\delta_{ij}b_{kp}\overline{\theta u_p}\,. \tag{A43}$$

If we neglect the nonlinear terms by taking

$$\gamma_{2,3,4,5} = 0\,, \tag{A44}$$

we recover equation (43a) of Paper I if we further call $f_1 \equiv 2c_6$ and $\gamma_1 \equiv c_7$. Substituting equation (A41) into equation (A16) above, we finally obtain the equation for $\overline{u_i\theta}$,

$$\frac{D}{Dt}\overline{u_i\theta} + \frac{\partial}{\partial x_j}(\overline{\theta u_iu_j} + \delta_{ij}\overline{p\theta}) = -\overline{u_iu_j}\frac{\partial T}{\partial x_j} - \left(1 - \frac{3}{4}\alpha_3\right)S_{ij}\overline{u_j\theta} - \left(1 - \frac{5}{4}\alpha_3\right)R_{ij}\overline{u_j\theta}$$
$$-f_1\tau^{-1}\overline{u_i\theta} + (1-\gamma_1)\lambda_i\overline{\theta^2} - \Pi_i^\theta(\mathrm{NL}) + \frac{1}{2}(\nu+\chi)\frac{\partial^2}{\partial x_j^2}\overline{u_i\theta} + C_i\,. \tag{A45}$$

APPENDIX B

THE ALGEBRAIC REYNOLDS STRESS MODEL

If one neglects the convective and diffusive terms, represented by the left sides of equation (A40) and (A45), these equations become algebraic and one can thus write the anisotropy tensor b_{ij} and the flux $\overline{u_i\theta}$ in a form that only entails the inversion of matrices. Such a model was used for example by Launder (1975). Physically, this corresponds to assuming that production P equals dissipation ϵ, as is clear from equation (A26) which becomes

$$P = \epsilon\,, \qquad P \equiv P_b + P_s\,, \tag{B1}$$

where the buoyancy and shear production terms are defined by

$$P_b = \lambda_i\overline{u_i\theta}\,, \qquad P_s = -b_{ij}S_{ij}\,. \tag{B2}$$

A better approximation can, however, be devised that, while accounting for the possibility that locally $P \neq \epsilon$, still leads to an algebraic model which of course can easily be reduced to the previous case by taking $P = \epsilon$. This approximation was originally suggested by Rodi (1984) and has been used extensively since then. We begin by considering equations (A23) and (A26), which we write in compact form as

$$\frac{D}{Dt}\overline{u_i u_j} + D_f(\overline{u_i u_j}) = \mathrm{RHS(A23)}\ , \tag{B3}$$

$$\frac{De}{Dt} + D_f(e) = \mathrm{RHS(A26)}\ , \tag{B4}$$

where RHS(A23, A26) means the right sides of equations (A23) and (A26) and $D_f(a)$ means the diffusion of the variable a (third-order moments). Introducing the dimensionless anisotropy tensor

$$a_{ij} = \frac{1}{e}\overline{u_i u_j} - \frac{2}{3}\delta_{ij} \equiv e^{-1}b_{ij}\ , \tag{B5}$$

equation (B3) becomes

$$e\frac{D}{Dt}a_{ij} + \frac{1}{e}\overline{u_i u_j}\left[\frac{De}{Dt} + D_f(e)\right] + \left[D_f(\overline{u_i u_j}) - \frac{1}{e}\overline{u_i u_j}D_f(e)\right] = \mathrm{RHS(A23)}\ . \tag{B6}$$

At this point we employ the approximation

$$eD_f(\overline{u_i u_j}) \approx \overline{u_i u_j}D_f(e) \tag{B7}$$

so that we can neglect the last parenthesis in the left side of equation (B6). Furthermore, using equations (B4), (A23), and (A26), we derive

$$e\frac{D}{Dt}a_{ij} + a_{ij}\left(P - \epsilon + \frac{1}{2}C_{ii}\right) = -2c_4^* e\tau^{-1}a_{ij} - \frac{8}{15}eS_{ij} + \beta_5 B_{ij} - (1-\alpha_1)\Sigma_{ij} - (1-\alpha_2)Z_{ij} + C_{ij} - \frac{1}{3}\delta_{ij}C_{kk} - \Pi_{ij}(\mathrm{NL})\ . \tag{B8}$$

Taulbee (1992) has recently shown that $Da_{ij}/Dt = 0$ is true only in the asymptotic case, namely for large values of the dimensionless parameter $\tau S \gg 1$, where $\tau = 2e/\epsilon$ is the characteristic time of turbulence, and S is the shear. It is clear that in many astrophysical settings of interest, for example in accretion disks, such a relation is not necessarily satisfied. Moreover, since the presence of buoyancy must also be accounted for, we extend Taulbee's suggestion by writing

$$S \to S_* = S(1 - R_f)^{1/2}\ , \tag{B9}$$

where the flux Richardson number is defined as the ratio P_b/P_s (see eq. [B2]).

In order to encompass arbitrary values of τS_*, it is suggested that one introduces the variable $a_{ij}/\tau S_*$ and take $D(a_{ij}/\tau S_*)/Dt = 0$. Using equation (19) of the text and the fact that $\tau = 2e/\epsilon$ as well as equation (15), one obtains

$$\frac{D\tau}{Dt} + \tau[e^{-1}D_f(e) + \epsilon^{-1}D_f(\epsilon)] = 2(c_2 - 1) - 2\frac{P}{\epsilon}(c_1^* - 1)\ , \tag{B10a}$$

$$c_1^* \equiv \frac{1}{P}(c_1 P_s + c_3 P_b)\ , \tag{B10b}$$

so that

$$\frac{1}{S_*}\frac{D}{Dt}(\tau S_*) = \frac{\tau}{S_*}\frac{D}{Dt}S_* + 2(c_2 - 1) - 2(c_1^* - 1)\frac{P}{\epsilon}\ . \tag{B10c}$$

Using equation (B10c) in equation (B8) yields equation (27a) of the text.

A similar treatment can be employed in the case of the convective flux $\overline{u_i\theta}$. In that case, it has been suggested by Gibson & Launder (1976) that the left side of equation (A45) be taken as

$$\frac{D}{Dt}\overline{u_i\theta} + D_f(\overline{u_i\theta}) = \frac{1}{2}\overline{u_i\theta}\left\{\frac{1}{e}\left[\frac{De}{Dt} + D_f(e)\right] + \frac{1}{\overline{\theta^2}}\left[\frac{D\overline{\theta^2}}{Dt} + D_f(\overline{\theta^2})\right]\right\}\ , \tag{B11}$$

which, upon using equations (A20) and (A26), becomes

$$\frac{D}{Dt}\overline{u_i\theta} + D_f(\overline{u_i\theta}) = B\tau^{-1}\overline{u_i\theta}\ , \tag{B12}$$

where

$$B \equiv \frac{P}{\epsilon} - 1 - \frac{1}{2}\epsilon^{-1}C_{ii} + c_\theta\left(\frac{P_\theta}{\epsilon_\theta} - 1 - Pe^{-1}\right)\ ,$$

$$P_\theta \equiv -\overline{u_i \theta} \frac{\partial T}{\partial x_i} + \frac{1}{2} C^\theta , \tag{B13}$$

the Peclet number being defined as

$$Pe = c_\epsilon \frac{e^{1/2}}{\chi} l_\epsilon \left| \frac{l_\epsilon^2}{\overline{\theta^2}} \frac{\partial^2 \overline{\theta^2}}{\partial x_j^2} \right|^{-1} . \tag{B14}$$

Substitution of equation (B12) into equation (A45) gives equation (28a) of the text.

APPENDIX C

THIRD-ORDER MOMENTS

The equations for the third-order moments are taken from Canuto (1992). We have

$$\left(\frac{D}{Dt} + \tau_3^{-1}\right)\overline{u_i u_j u_k} = -(\overline{u_i u_j u_l}\, U_{k,l} + \text{perm.}) - \left(\overline{u_i u_l} \frac{\partial}{\partial x_l} \overline{u_j u_k} + \text{perm.}\right) + (1 - c_{11})(\lambda_i \overline{\theta u_j u_k} + \text{perm.}) - \frac{2}{3\tau} (\delta_{ij} \overline{q^2 u_k} + \text{perm.}) , \tag{C1}$$

$$\left(\frac{D}{Dt} + \tau_3^{-1}\right)\overline{u_i u_j \theta} = \overline{u_i u_j u_k}\, \beta_k - (\overline{u_i u_k \theta} U_{j,k} + \overline{u_j u_k \theta} U_{i,k}) - \left(\overline{u_i u_k} \frac{\partial}{\partial x_k} \overline{\theta u_j} + \overline{u_j u_k} \frac{\partial}{\partial x_k} \overline{\theta u_i} + \overline{\theta u_k} \frac{\partial}{\partial x_k} \overline{u_i u_j}\right)$$
$$+ \frac{2}{3} c_{11} \delta_{ij} \lambda_k \overline{\theta^2 u_k} + \tau^{-1} c_* \delta_{ij} \overline{q^2 \theta} + (1 - c_{11})(\lambda_i \overline{\theta^2 u_j} + \lambda_j \overline{\theta^2 u_i}) , \tag{C2}$$

$$\left(\frac{D}{Dt} + \tau_3^{-1} + 2\tau_\theta^{-1}\right)\overline{u_i \theta^2} = 2\overline{\theta u_i u_j}\, \beta_j - \overline{\theta^2 u_j}\, U_{i,j} - 2\overline{\theta u_j} \frac{\partial}{\partial x_j} \overline{\theta u_i} + (1 - c_{11})\lambda_i \overline{\theta^3} - \overline{u_i u_j} \frac{\partial}{\partial x_j} \overline{\theta^2} , \tag{C3}$$

$$\left(\frac{D}{Dt} + \frac{c_{10}}{c_8} \tau_3^{-1}\right)\overline{\theta^3} = 3\overline{\theta^2 u_j}\, \beta_j - 3\overline{\theta u_j} \frac{\partial}{\partial x_j} \overline{\theta^2} + \chi \frac{\partial^2}{\partial x_l^2} \overline{\theta^3} , \tag{C4}$$

where β_i has been defined in equation (3c) of the text, and

$$U_{i,j} \equiv \frac{\partial U_i}{\partial x_j} , \qquad \tau = \frac{2e}{\epsilon} , \qquad \tau_3 \equiv \frac{\tau}{2c_8} , \qquad \tau_\theta = \frac{\overline{\theta^2}}{\epsilon_\theta} , \tag{C5}$$

In the stationary case, it has been customary for many years to approximate equations (C1)–(C3) by

$$\overline{u_i u_j u_k} \to -\tau_3 \overline{u_k u_l} \frac{\partial}{\partial x_l} \overline{u_i u_j} , \qquad \overline{u_i u_j \theta} \to -\tau_3 \overline{u_j u_k} \frac{\partial}{\partial x_k} \overline{\theta u_i} , \qquad \overline{u_i \theta^2} \to -\tau_3(1 + 2\tau_3 \tau_\theta^{-1})\overline{u_i u_j} \frac{\partial}{\partial x_j} \overline{\theta^2} . \tag{C6}$$

If the degree of anisotropy is assumed to be small,

$$b_{ij} = \overline{u_i u_j} - \tfrac{2}{3} e\, \delta_{ij} \approx 0 , \tag{C7}$$

one further has

$$\overline{u_i u_j u_k} \to -A\nu_t \frac{\partial}{\partial x_k} \overline{u_i u_j} , \qquad \overline{u_i u_j \theta} \to -A\nu_t \frac{\partial}{\partial x_j} \overline{\theta u_i} , \qquad \overline{u_i \theta^2} \to -B\nu_t \frac{\partial}{\partial x_i} \overline{\theta^2} , \tag{C8}$$

with $3c_8 c_\nu A \equiv 2$ and $B \equiv A(1 + 2\tau_3 \tau_\theta^{-1})$; the turbulent viscosity ν_t is given by equation (32b). Equations (C6) and (C8) represent the *down-gradient, diffusive approximation to the third-order moments.*

As discussed in Paper I, equations (C8) lead to incorrect results: for example, planetary boundary layer data show that

$$\overline{w^3} > 0 \quad \text{and} \quad \frac{\partial}{\partial z} \overline{w^2} > 0 , \tag{C9}$$

which contradict the first part of equation (C8). Recently, the system of equations (C1)–(C4) has been inverted exactly in the case where there is no mean flow and the variables depend only on z (Canuto et al. 1994). The principal result is that all third-order moments exhibit a universal structure: they all are a linear combination of the gradients of all the second-order moments. For example,

$$\overline{q^2 w} = D_1 \frac{\partial}{\partial z} \overline{w^2} + D_2 \frac{\partial}{\partial z} \overline{q^2} + D_3 \frac{\partial}{\partial z} \overline{w\theta} + D_4 \frac{\partial}{\partial z} \overline{\theta^2} , \tag{C10}$$

where the turbulent diffusivities D_k have the general structure

$$D \approx a\nu_t + b\overline{w\theta} , \tag{C11}$$

indicating that the turbulent diffusivities are contributed not only by the mechanical part $\nu_t \sim wl$, but also by buoyancy.

While the approximations

$$U_i = 0\,, \qquad \frac{\partial}{\partial x_i} \to \frac{\partial}{\partial z} \tag{C12}$$

can be made in an ensemble average approach, they are evidently incorrect within the context of an LES where all the components of U_i and all the space dependence must be kept, for U_i does not represent the external field but the one describing the large scales. This makes the inversion of equations (C1)–(C4) a harder task, even with the help of symbolic algebra. Until that is done, we suggest the use of a temporary intermediate solution between equations (C6) and (C10). The idea is to begin with the down-gradient approximation as a zero-order solution ($n = 0$) to be substituted back into all the third-order moments appearing on the right sides of equations (C1)–(C4). This will provide, without the need of a matrix inversion, a new set of third-order moments

$$\overline{u_i u_j u_k}\,, \qquad \overline{u_i u_j \theta}\,, \qquad \overline{u_i \theta^2}\,. \tag{C13}$$

The calculation cannot, however, be stopped at this stage. In fact, one can notice that at this level of approximation, one would not recover all the terms appearing in equation (C10): for example, the first third-order moment in equation (C13) would not contain the last term in equation (C10), namely the gradient of the temperature variance, as one observes by inspecting the components of the zero-order expression for $\overline{u_i u_j \theta}$. On the other hand, if one goes one step further, such a dependence is recovered.

Formally, the procedure can be written as follows: define

$$A_{ijk} \equiv \overline{u_i u_j u_k}\,, \qquad B_{ij} \equiv \overline{\theta u_i u_j}\,, \qquad C_i \equiv \overline{u_i \theta^2}\,, \qquad D \equiv \overline{\theta^3}\,. \tag{C14}$$

Then we have from equation (C1),

$$A_{ijk}^{n+1} = A_{ijk}^0 - \tau_3(A_{ijl}^n U_{k,l} + \text{perm.}) + (1 - c_{11})\tau_3(\lambda_i B_{jk}^n + \text{perm.}) - (1/3c_8)(\delta_{ij} A_{ppk}^n + \text{perm.})\,, \tag{C15}$$

where the zeroth-order moments are defined as

$$-\tau_3^{-1} A_{ijk}^0 \equiv \overline{u_i u_l}\,\frac{\partial}{\partial x_l}\,\overline{u_j u_k} + \text{perm.}\,, \tag{C16}$$

$$-\tau_3^{-1} B_{ij}^0 \equiv \overline{u_i u_k}\,\frac{\partial}{\partial x_k}\,\overline{\theta u_j} + \overline{u_j u_k}\,\frac{\partial}{\partial x_k}\,\overline{\theta u_i} + \overline{\theta u_k}\,\frac{\partial}{\partial x_k}\,\overline{u_i u_j}\,, \tag{C17}$$

$$-\tau_3^{-1}(1 + 2\tau_3 \tau_\theta^{-1}) C_i^0 \equiv 2\overline{\theta u_j}\,\frac{\partial}{\partial x_j}\,\overline{\theta u_i} + \overline{u_i u_j}\,\frac{\partial}{\partial x_j}\,\overline{\theta^2}\,, \tag{C18}$$

$$-\frac{c_{10}}{c_8}\,\tau_3^{-1} D^0 \equiv 3\overline{\theta u_j}\,\frac{\partial}{\partial x_j}\,\overline{\theta^2}\,. \tag{C19}$$

Analogously, equation (C2) becomes with $c_* = 0$,

$$B_{ij}^{n+1} = B_{ij}^0 + \beta_k \tau_3 A_{ijk}^n - \tau_3(B_{ik}^n U_{j,k} + B_{jk}^n U_{i,k}) + \tfrac{2}{3} c_{11}\,\delta_{ij} \tau_3 \lambda_k C_k^n + (1 - c_{11})\tau_3(\lambda_i C_j^n + \lambda_j C_i^n)\,. \tag{C20}$$

Equation (C3) becomes

$$C_i^{n+1} = C_i^0 + T[2\beta_j B_{ij}^n - C_j^n U_{i,j} + (1 - c_{11})\lambda_i D^n]\,, \tag{C21}$$

where

$$T \equiv \tau_3(1 + 2\tau_3/\tau_\theta)^{-1}\,. \tag{C22}$$

Finally, equation (C4) becomes

$$D^{n+1} = D^0 + (3c_8/c_{10})\tau_3 \beta_i C_i^n\,. \tag{C23}$$

The remaining third-order moments are determined by the relations

$$\overline{pu_i} = -C_p^w \overline{q^2 u_i}\,, \qquad \overline{p\theta} = -C_p^\theta \overline{q^2 \theta}\,. \tag{C24}$$

APPENDIX D

THE PARAMETERS

The constants are defined as follows:

1. Equations (27)–(28):

$$c_4^* = c_4 + 1 - F^{1/2}\,, \qquad c_4 = 1 + 6.22F^2(1 - F)^{3/4}\,, \qquad f_1 = 7.5\,,$$

$$\alpha_1 = 6\alpha_5\,, \qquad 3\alpha_2 = 2(2 - 7\alpha_5)\,, \qquad \alpha_3 = \tfrac{4}{5}\,, \qquad \alpha_4 = \tfrac{3}{10}\,, \qquad 10\alpha_5 = 1 + \tfrac{4}{5}F^{1/2}\,, \qquad \alpha_6 = \tfrac{1}{10}\,,$$

$$F = 1 + 9\mathrm{II} + 27\mathrm{III}\,. \tag{D1}$$

The invariants II and III are defined below (eq. [D8]).

2. Constants entering the third-order moments (Appendix C):

$$c_8 = 8 , \quad c_{10} = 4 , \quad c_{11} = \tfrac{1}{5} , \quad c_* = 0 , \qquad C_p^w = C_p^\theta = \tfrac{1}{5} . \tag{D2}$$

3. The functions γ's entering equation (A41a) and (A42) (Shih & Shabbir 1992):

$$9\gamma_1 \equiv 6r^2 - 10 - \beta_5(r^2 - 1)^{-1}[18(r^2 + 1)rb + r^2(7 - 15r^2) + 36\text{II} - 10] , \tag{D3}$$

$$9(r^2 - 1)\text{II}\gamma_2 \equiv r^2(3\text{II} + 14) - (3\text{II} + \tfrac{7}{2}) - \tfrac{21}{2}r^4 - \beta_5[r^2(12\text{II} + 20) + 108\text{II}rb + 108\text{II}^2 - 21\text{II} - 5 - 15r^4(1 + 3\text{II})] , \tag{D4}$$

$$\gamma_3 \equiv -1 + \tfrac{1}{2}\beta_5(r^2 - 1)^{-1}(12rb - 5r^2 + 12\text{II} - 1) , \tag{D5}$$

$$\gamma_4 \equiv -\tfrac{3}{2}\beta_5 , \qquad 6\text{II}\gamma_5 \equiv 7 + \beta_5(36\text{II} - 10) , \tag{D6}$$

where

$$2r^2 \equiv \overline{\theta u_i}\,\overline{\theta u_i}(\overline{\theta^2}e)^{-1} , \qquad 2rb \equiv (\overline{\theta^2}e)^{-1}\overline{\theta u_i}\,\overline{\theta u_j}b_{ij} , \tag{D7}$$

$$-8e^2\text{II} \equiv b_{ij}b_{ij} , \qquad 24e^3\text{III} \equiv b_{ij}b_{jk}b_{ki} . \tag{D8}$$

Using data from a buoyant plume experiment, Shih & Shabbir (1992) have determined that the value of β_5 is approximately 0.6 which would correspond to $c_5 \equiv 1 - \beta_5 = 0.4$, a value close to 0.3 suggested in Paper I, equation (44d). In the same work, the authors have also shown that γ_1 is almost constant (~ 0.42), while $\gamma_{2,3,4,5}$ are all negative, with values ranging as follows:

$$1.91 < |\gamma_2| < 2.82 , \quad 0.28 < |\gamma_3| < 0.7 , \qquad 0.81 < |\gamma_4| < 2.76 , \quad 0.95 < |\gamma_5| < 3.15 . \tag{D9}$$

4. The functions β_5, β_7, and β_9 entering the nonlinear term Δ_{ij}^k, equation (A29) (Shih & Shabbir 1992), are given by

$$\beta_5(6\text{II} - 10r^2 - 36\text{II}rb) = -(12\text{II} + 7)r^2 , \tag{D10}$$

$$\text{II}\beta_7 = -\tfrac{7}{6} + \beta_5(\tfrac{5}{3} - \text{II}) , \tag{D11}$$

$$-\beta_9 = \beta_5 + \beta_7 . \tag{D12}$$

5. Equation (19):

$$c_1 = 1.44 , \qquad c_2 = 1.83 , \qquad c_3 = 1.82 . \tag{D13}$$

6. Equation (20):

$$\frac{1}{2}c_\theta = c_\epsilon \frac{\text{Ko}}{\text{Ba}} , \qquad c_\epsilon = \pi\left(\frac{2}{3\text{Ko}}\right)^{3/2} , \qquad \text{Ko} = 1.6 \pm 0.02 , \qquad \text{Ba} = 1.34 \pm 0.02 , \tag{D14}$$

where the value of Ko and Ba are taken from Andreas (1987).

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